2:1 Paper Chain

Teacher Notes

Math Milestones™

Central math concepts

Based on interviews with middle-grades students,† education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

In particular, task 2:1 involves addition and subtraction and the relationship between them. The mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between addition and subtraction can play

2:1 Avi made a paper chain. Then Avi added 29 more links to the paper chain. Now there are 52 links in the paper chain. How many links were in the paper chain before?

Answer

There were 23 links in the paper chain before.

Click here for a student-facing version of the task.

Refer to the Standards

2.OA.A.1, 2.NBT.B.5; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application, Procedural skill and fluency

Additional notes on the design of the task

- The calculation involved in the task, 52 - 29, is a two-digit addition problem which belongs to the fluency target in grade 2; see CCSS 2.NBT.B.5 and the Teacher Notes for task **2:3 Fluency within 100 (Add/Subtract)**.
- The first sentence, "Avi made a paper chain," does not refer explicitly to one of the quantities in the situation, which is the number of links that were in the paper chain to begin with. Nevertheless, as soon as the chain was made, that quantity was present in the situation. The quantity is eventually named explicitly by the last sentence in the task (the sentence ending with a question mark).

out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:[‡]

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems.

In particular, the situation type in task 2:1 is called "Add To with Start Unknown." It is an Add To situation because Avi adds links to the chain; and more specifically, the situation is "Add To with Start Unknown" because the initially unknown quantity is how many links were in the chain to begin with.

During the primary grades, students work with all situation types and all variations in the known and unknown quantities, with quantities given as whole numbers. In the upper-elementary grades, these understandings of addition and subtraction are extended into working with fractional quantities. Although the algorithms for performing calculations with fractions are different from those for performing base-ten calculations with whole numbers, the underlying meanings and uses of addition and subtraction are the same regardless of whether the numbers involved are whole numbers, fractions, decimals, or even variables.

Word problems vary considerably in the uses to which they put addition and subtraction, and they also vary in the complexity of the calculation required to obtain a final numerical answer. The overall challenge of a word problem depends on both the situational complexity and the computational complexity. The calculation in task 2:1 involves calculating a difference of two two-digit numbers in a case that also involves regrouping. Calculations of this kind are new to students in grade 2; students work toward fluency with these calculations throughout the year. Students for whom the calculation 52 - 29 is time-consuming and/or effortful may need to be redirected to the context after obtaining the result 52 - 29 = 23, so as to relate the numbers in this equation to the context and answer the question in the task.

The difference 52 - 29 could be calculated in many ways using place value, properties of operations, and/or the relationship between addition and subtraction. A student could use a pencil and paper method based on tens and ones (see Teacher Notes for task **2:3 Fluency within 100 (Add/Subtract)**). A student could calculate mentally in ways such as in the following examples:

- 52 - 29 = 52 - 30 + 1 = 22 + 1 = 23.
- 29 + **1** = 30, 30 + **20** = 50, 50 + **2** = 52, so 29 + **23** = 52.

Such mental calculation methods should be discussed and compared in connection with corresponding written equations.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:1? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:1? In what specific ways do they differ from 2:1?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:1 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 2, p. 9 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: adding and/or subtracting with two-digit numbers; and working with the count sequence within 100.

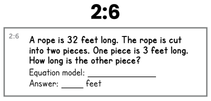
Extending the task

How might students drive the conversation further?

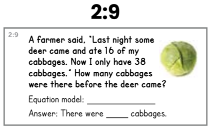
- Students could represent the quantities in the situation in two ways, by creating a diagram (such as a tape diagram, for example) and by creating a situation equation (such as ? + 29 = 52), then identify correspondences between the equation and the diagram.
- Students could check their difference by adding.
- Students could imagine that the chain got torn in two. Assuming one link was ruined and thrown away, then what are two possible numbers for the number of links in the two smaller chains?

Related Math Milestones tasks

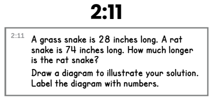
2:6


2:6 A rope is 32 feet long. The rope is cut into two pieces. One piece is 3 feet long. How long is the other piece?
Equation model: ____________
Answer: ____ feet

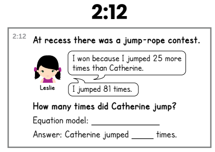
2:9


2:9 A farmer said, "Last night some deer came and ate 16 of my cabbages. Now I only have 38 cabbages." How many cabbages were there before the deer came?
Equation model: ____________
Answer: There were ____ cabbages.

2:11

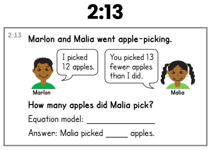

2:11 A grass snake is 28 inches long. A rat snake is 74 inches long. How much longer is the rat snake?
Draw a diagram to illustrate your solution. Label the diagram with numbers.

2:12


2:12 At recess there was a jump-rope contest.

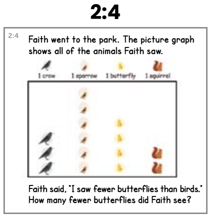
How many times did Catherine jump?
Equation model: ____________
Answer: Catherine jumped ____ times.

2:13


2:13 Marlon and Malia went apple-picking.

How many apples did Malia pick?
Equation model: ____________
Answer: Malia picked ____ apples.

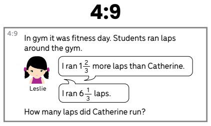
2:4


2:4 Faith went to the park. The picture graph shows all of the animals Faith saw.

Faith said, "I saw fewer butterflies than birds."
How many fewer butterflies did Faith see?

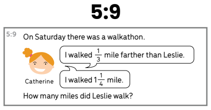
Other word problems and their situation types in grade 2 are as follows: tasks **2:6 Cutting a Rope**, *Put Together with One Addend Unknown*, in a context involving length; **2:9 Disappearing Cabbages**, *Take From with Start Unknown*; **2:11 Grass Snake vs. Rat Snake**, *Compare with Difference Unknown ('how many more'/'how much longer' language)*, in a context involving length; **2:12 Jump-Rope Contest**, *Compare with Smaller Quantity Unknown ('more' language)*; **2:13 Apple-Picking**, *Compare with Bigger Quantity Unknown ('fewer' language)*; and **2:4 Animals in the Park**, combination of *Put Together/Take Apart with Total Unknown and Compare with Difference Unknown ('how many fewer' language)*.

4:9


4:9 In gym it was fitness day. Students ran laps around the gym.

How many laps did Catherine run?

5:9


5:9 On Saturday there was a walkathon.

How many miles did Leslie walk?

In later grades, tasks **4:9 Fitness Day** and **5:9 Walkathon** are addition/subtraction word problems of situation type *Compare with Smaller Quantity Unknown*.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones](#).

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:2 Place Value to Hundreds

Teacher Notes

Central math concepts

The place value system derives its power from several notable features:

1. **The base-ten units.** Because base-ten units increase exponentially in size (ones, tens, hundreds, thousands, and so on, increasing at each place by a factor of ten), enormous quantities can be represented with few digits. For example, consider the diameter of the Sun, approximately 865,000 miles. To write this number using tally marks would require about a hundred sheets of paper…but only six digits were required to represent the number using base-ten units.
2. **Flexible bundling and unbundling of the base-ten units.** Because of their compatible relative sizes, base-ten units can be bundled or unbundled into other base-ten units. Ten ones make a larger unit called "a ten." Ten tens make a larger unit called "a hundred." And ten hundreds make a larger unit called "a thousand." It follows, then, that a hundred ones make a hundred, a thousand ones make a thousand, and a hundred tens make a thousand. Bundling and unbundling are central ideas in base-ten calculation algorithms. Rods, flats, and cubes provide concrete illustrations of base-ten units and how they can be bundled and unbundled.
3. **Positional notation.** The location of each digit in a multi-digit number corresponds to a base-ten unit, with the units ordered by convention from right to left in increasing order of size. The digit in a given place tells how many copies of the corresponding unit are in the number. For example, the quantity 9 hundreds and 8 ones is written as 908.

These three aspects of place value work together, and understanding place value entails coordinating all three aspects. For example, working with rods, flats, and cubes won't by itself teach place value, because these manipulatives aren't connected to positional notation.

As noted in the relevant *Progression* document,[†] the place value system is a single system that includes both whole numbers and decimals:

> The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals. (p. 2)

The place value system is often considered one of the most impressive inventions in the history of mathematics. For the most part however, students are learning place value not as a standalone topic of study, but rather as a necessity for learning to calculate sums and differences of multi-digit numbers with understanding.

2:2 **(1) True or false?**

(a) 2 hundreds + 3 ones > 5 tens + 9 ones

(b) 9 tens + 2 hundreds + 4 ones < 924

(c) 456 < 5 hundreds

(2) Write the number that makes each statement true.

(a) 7 ones + 5 hundreds = _____

(b) 14 tens = _____

(c) 90 + 300 + 4 = _____

Answer

(1) (a) True. (b) True. (c) True.
(2) (a) 507. (b) 140. (c) 394.

Click here for a student-facing version of the task.

Refer to the Standards

2.NBT.A; MP.1, MP.6, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- Task 2:2 is designed to target conceptual understanding, even though it only asks for brief answers rather than extended writing or other language demands. Teachers can also question students about the thinking that led to their answers, individually or in a group setting (and students can question each other).

The *Progression* document includes many additional useful points relevant to the teaching and learning of place value in grade 2, including the following:

- "Unlike the decade words, the hundred words indicate base-ten units. For example, it takes interpretation to understand that "fifty" means five tens, but "five hundred" means almost what it says ("five hundred" rather than "five hundreds"). Even so, this doesn't mean that students automatically understand 500 as 5 hundreds; they may [at first] only think of it as the number said after 499 or reached after 500 counts of 1" (p. 8).
- "Comparing magnitudes of two-digit numbers uses the understanding that 1 ten is greater than any amount of ones represented by a one-digit number. Comparing magnitudes of three-digit numbers uses the understanding that 1 hundred (the smallest three-digit number) is greater than any amount of tens and ones represented by a two-digit number" (p. 8).

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: reasoning about numbers within 100 based on their size in terms of tens and ones; and fluency with the count sequence within 1000.

Extending the task

How might students drive the conversation further?

- Students could create their own versions of task 2:2 by keeping the digits the same but mixing up the place value units. For example, part (2) (a) could be mixed up to read "7 hundreds + 5 ones = _______." Students could trade their problems with a partner and check each other's answers.
- Could it lead to confusion and trouble in society if some schools taught their students that 924 means 9 hundreds, 2 tens, and 4 ones while other schools taught that 924 means 9 ones, 2 hundreds, and 4 tens? Students could discuss the value of having society agree on a single way of writing and reading numbers.

Related Math Milestones tasks

2:7

2:7 (1) Write the number that makes the statement true.
6 hundreds + 3 tens + 4 ones
= 5 hundreds + _____ tens + 4 ones.
(2) How do you know your statement is true?
(3) Look for connections between your statement and this subtraction problem. What connections can you see?

$$\begin{array}{r} \overset{5\,13}{634} \\ -\,482 \\ \hline 152 \end{array}$$

2:10

2:10 Check the subtraction by adding.
946 − 678 = 268

3:14

3:14 Write the sums and differences.

With pencil and paper

$$\begin{array}{r} 351 \\ +\,472 \\ \hline \end{array} \quad \begin{array}{r} 264 \\ +\,438 \\ \hline \end{array} \quad \begin{array}{r} 625 \\ -\,261 \\ \hline \end{array} \quad \begin{array}{r} 831 \\ -\,444 \\ \hline \end{array}$$

Mentally
800 − 300
240 + 540
365 − 165
612 − 13

1:2

1:2 True or false?
6 tens + 4 ones < 4 ones + 7 tens

7 ones + 5 tens = _____

1:8

1:8 90 − 40 = _______
9 apples − 4 apples = _______ (number) _______ (unit)
9 cups − 4 cups = _______ (number) _______ (unit)
9 tens − 4 tens = _______ (number) _______ (unit)

Task **2:7 Subtraction Regrouping** relates a calculation procedure to fundamental concepts of place value. Task **2:10 Three Digit Addition/**

Additional notes on the design of the task (continued)

- Place value units of hundreds, tens, and ones appear in several different orders so that relating the named quantities to base-ten numerals in positional notation is part of the task.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:2? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:2? In what specific ways do they differ from 2:2?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:2 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2015, March 6). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5, Number and Operations in Base Ten*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Subtraction emphasizes the relationship between addition and subtraction in a three-digit calculation.

In later grades, task **3:14 Fluency within 1000 (Add/Subtract)** is a fluency task involving calculation of three-digit sums and differences.

In earlier grades, task **1:2 Tens and Ones** is an analogue of task 2:2 that involves base ten units of tens and ones. Task **1:8 Subtracting Units** emphasizes unit thinking in two-digit subtraction of multiples of ten.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:3 Fluency within 100 (Add/Subtract)

Teacher Notes

Math Milestones™

Central math concepts

Task 2:3 focuses on fluency in adding and subtracting within 100 (CCSS 2.NBT.B.5). Because such problems are a special case of adding and subtracting within 1000,† it is worth discussing first the three-digit methods that students use in grade 2. For three-digit problems, students in grade 2 learn to use an efficient, accurate, and generalizable strategy that can develop further to meet the procedural expectations in grade 3 and grade 4. For example, pages 9 and 10 of the *Progression* document show two calculation methods for three-digit addition. Images of these methods are presented below.

Addition: Recording newly composed units in separate rows

```
  278      278      278      278
 +147     +147     +147     +147
 ----     ----     ----     ----
           300      300      300
                    110      110
                              15
                            ----
                             425
```

The computation shown proceeds from left to right, but could have gone from right to left. Working from left to right has two advantages: Many students prefer it because they read from left to right; working first with the largest units yields a closer approximation earlier.

Addition: Recording newly composed units in the same row

```
  278      278      278      278
 +147     +147     +147     +147
 ----     ---1-    -1-1-    -1-1-
            5       25      425
```

	Add the ones, $8+7$, and record these 15 ones with 1 on the line in the tens column and 5 below in the ones place.	Add the tens, $7+4+1$, and record these 12 tens with 1 on the line in the hundreds column and 2 below in the tens place.	Add the hundreds, $2+1+1$, and record these 4 hundreds below in the hundreds column.

Digits representing newly composed units are placed below the addends, on the line. This placement has several advantages. Each two-digit partial sum (e.g., "15") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 5) rather than "write the 5 and carry the 1" (write 5, then 1).

The *Progression* document notes that "The first written method is a helping step variation that generalizes to all numbers in base ten but becomes impractical because of writing so many zeros. Students can move from this method to the second method (or another compact method) by seeing how the steps of the two methods are related. Some students might make this transition in Grade 2, some in Grade 3, but all need to make it by Grade 4 where fluency requires a more compact method."

2:3 Write the sums and differences.

```
  36     72     64     82
+ 45   - 17   + 27   - 55
----   ----   ----   ----
```

Answer

81, 55, 91, 27.

Click here for a student-facing version of the task.

Refer to the Standards

2.NBT.B.5; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

The task does not require students to show their work, but looking at students' steps can show where they may have made a careless mistake.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:3? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 2:3? In what specific ways do they differ from 2:3?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:3 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

To return to the case of two-digit addition, the *Progression* document notes that after working with three-digit sums, “the type of problems required for fluency in Grade 2 seem easy, e.g., 28 + 47 requires only composing a new ten from ones. This is now easier to do without drawings: one just records the new ten before it is added to the other tens or adds it to them mentally.” Similarly, for subtraction, after working with three-digit differences requiring two, one, and no decompositions, “the objectives for fluency at this grade are easy: focusing within 100 just on the two cases of one decomposition (e.g., 73 - 28) or no decomposition (e.g., 78 - 23) without drawings.” Thus, treating two-digit problems as a special case of three-digit problems is not only mathematically accurate, but also helpful to students.

Conversely, as noted on pages 9 and 10 of the *Progression* document, “Spending a long time on subtraction within 100 can stimulate students to count on or count down, [both of which] are considerably more difficult with numbers above 100.” To be sure, “If [counting-on and adding-on methods] arise from students, they should be discussed. But the major focus for addition within 1000 needs to be on methods such as those [shown in the two images] that are simpler for students and lead toward fluency (e.g., [as in the first image]) or are sufficient for fluency (e.g., [as in the second image]).”

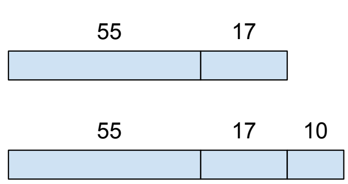
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value concepts; and single-digit sums and differences.

Extending the task

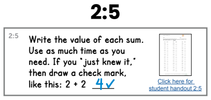
How might students drive the conversation further?

- Checking differences by adding can offer additional procedural practice and reinforce the relationship between addition and subtraction ($C - A$ is the unknown factor in $A + \square = C$).
- Students could compare the results 72 - 17 = 55 and 82 - 55 = 27. Do these two results make sense together? (See the figure for one approach to discussing this question.)

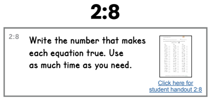


Related Math Milestones tasks

2:5



2:8

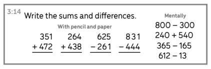


Tasks **2:5 Sums of Single-Digit Numbers** and **2:8 Fluency within the Addition Table** involve the single-digit sums and related differences upon which multi-digit addition and subtraction methods are built. Several of the word problems in grade 2 involve two-digit sums and differences (see the Map of Addition and Subtraction Situations in K–2 Math Milestones).

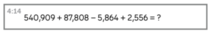
† Common Core Standards Writing Team. (2015, March 6). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5, Number and Operations in Base Ten.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona, p. 9.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

3:14

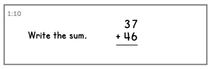


4:14

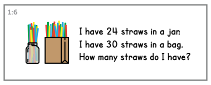


In later grades, task **3:14 Fluency within 1000 (Add/Subtract)** continues the fluency progression for multi-digit addition and subtraction, and task **4:14 Fluency with Multi-Digit Sums and Differences** marks the culmination of this progression.

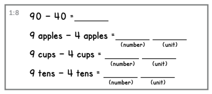
1:10



1:6



1:8



In earlier grades, tasks **1:10 Two-Digit Addition** and **1:6 Two Groups of Straws** involve the sum of two two-digit numbers. Task **1:8 Subtracting Units** involves subtracting with place value units of tens.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:4 Animals in the Park

Teacher Notes

Central math concepts

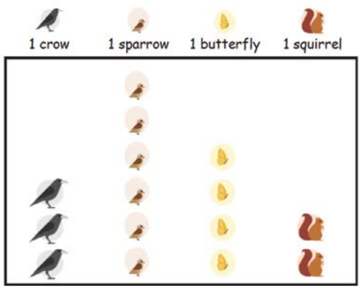
Students' data work in the primary grades involves representing and interpreting categorical data, which is data that arises from classifying or sorting into categories.[†] In a picture graph like the one shown in task 2:4, the small pictures of animals stacked in columns are the individual data points. Interpreting such a chart involves grasping the correspondence between a data point (picture of an animal) and the particular fact it represents (seeing an animal of that kind).

Students' analysis of categorical data connects directly to their uses of addition and subtraction to solve problems in context. In grade 2, when students "solve simple put-together, take-apart, and compare problems using information presented in a bar graph" (2.MD.D.10), they are also using "addition and subtraction ... to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions" (2.OA.A.1). More generally, there are close connections in every elementary grade between students' data work and their expanding use of numbers and operations in context; see Table 1, p. 4 of the *Progression* document for a list of these connections in grades K–5.

In grade 2, students draw picture graphs in which each picture represents one object, and they draw bar graphs with a single-unit scale. Drawing a picture graph or a bar graph involves identifying quantities in the situation, specifying units of measure, and attending to precision. To support students in creating picture graphs and bar graphs, grid paper is useful. As noted in the *Progression* document (p. 7), "When drawing picture graphs on grid paper, the pictures representing the objects should be drawn in the squares of the grid paper." For bar graphs, "the tick marks on the count scale should be drawn at intersections of the gridlines. The tops of the bars should reach the respective gridlines of the appropriate tick marks." For both kinds of graphs, as suggested in the *Progression* document (p. 3), "a template can be superimposed on the grid paper, with blanks provided for the student to write in the graph title, scale labels, category labels, legend, and so on."

As the *Guidelines for Assessment and Instruction in Statistics Education Report* notes, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning."[‡] Thus as the *Progression* document notes, "students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the context they represent" (p. 3). That said, "students do not have to generate the data every time they work on making bar graphs and picture graphs. That would be too time-consuming. After some experiences in generating the data, most work in producing bar graphs and picture graphs can be done by providing students with data sets" (p. 7). In task 2:4, the picture graph is provided for the student.

2:4 Faith went to the park. The picture graph shows all of the animals Faith saw.



Faith said, "I saw fewer butterflies than birds." How many fewer butterflies did Faith see?

Answer

Faith saw 5 fewer butterflies than birds.

Click here for a student-facing version of the task.

Refer to the Standards

2.MD.D.10; MP.4, MP.6. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

- The task blends two situation types, *Put Together with Total Unknown* and *Compare with Difference Unknown* ('fewer' language).

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: comparing numbers; adding and subtracting single-digit numbers; and using addition and subtraction to solve problems in context.

Extending the task

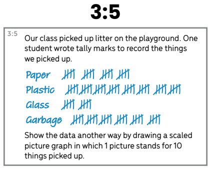
How might students drive the conversation further?

- Students could create a bar graph of the same data and compare the way it represents the data to the way the picture graph represents the data. Are there advantages to each?
- Students could recategorize the data in three categories: Birds, Butterflies, and Squirrels. What would the bar graph look like for these categories?
- Students could ask questions similar to those in the task to analyze another set of data they have collected in the classroom or in their community.

Related Math Milestones tasks

See the Map of Addition and Subtraction Situations in K–2 Math Milestones to find tasks in grades K, 1, and 2 that cover the various addition and subtraction situation types.

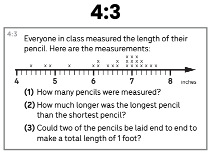
3:5


3:5 Our class picked up litter on the playground. One student wrote tally marks to record the things we picked up.

Paper
Plastic
Glass
Garbage

Show the data another way by drawing a scaled picture graph in which 1 picture stands for 10 things picked up.

4:3

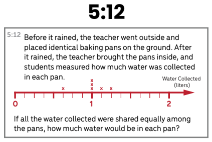

4:3 Everyone in class measured the length of their pencil. Here are the measurements:

(1) How many pencils were measured?

(2) How much longer was the longest pencil than the shortest pencil?

(3) Could two of the pencils be laid end to end to make a total length of 1 foot?

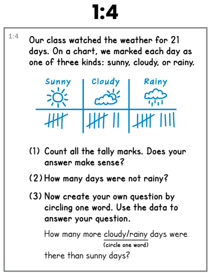
5:12


5:12 Before it rained, the teacher went outside and placed identical baking pans on the ground. After it rained, the teacher brought the pans inside, and students measured how much water was collected in each pan.

If all the water collected were shared equally among the pans, how much water would be in each pan?

In later grades, task **3:5 Playground Cleanup** involves creating a scaled picture graph. Tasks **4:3 Pencil Data** and **5:12 Rain Measurements** involve problem solving based on measurement data displayed in line plots.

1:4


1:4 Our class watched the weather for 21 days. On a chart, we marked each day as one of three kinds: sunny, cloudy, or rainy.

(1) Count all the tally marks. Does your answer make sense?

(2) How many days were not rainy?

(3) Now create your own question by circling one word. Use the data to answer your question.

How many more cloudy/rainy (circle one word) days were there than sunny days?

K:14


K:14 Are there more land animals or more sea animals?

In earlier grades, task **1:4 Analyzing Weather Data** involves asking and answering questions about categorical data organized in a tally chart. Task **K:14 Animals from Land and Sea** involves a comparison based on classifying into categories.

Additional notes on the design of the task (continued)

- If a student says, "Faith saw 2 fewer butterflies than birds," then the student might not have thought to combine the crows and the sparrows into the category (or unit) "birds." In that case, it is advisable to respond to the student by emphasizing what the student did correctly. The student correctly compared the number of butterflies Faith saw to the number of sparrows Faith saw.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:4? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:4? In what specific ways do they differ from 2:4?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:4 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2011, June 20). *Progressions for the Common Core State Standards in Mathematics (draft): K–3, Categorical Data; Grades 2–5, Measurement Data*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ The Guidelines for Assessment and Instruction in Statistics Education Report was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:5 Sums of Single-Digit Numbers

Teacher Notes

Math Milestones™

Central math concepts

Task 2:5 draws on memory and fluency. The table illustrates how the 66 brief problems in task 2:5 map to the addition table.

+	0	1	2	3	4	5	6	7	8	9
0										
1										✔
2			✔	✔	✔	✔	✔	✔	✔	✔
3			✔	✔	✔	✔	✔	✔	✔	✔
4			✔	✔	✔	✔	✔	✔	✔	✔
5			✔	✔	✔	✔	✔	✔	✔	✔
6			✔	✔	✔	✔	✔	✔	✔	✔
7			✔	✔	✔	✔	✔	✔	✔	✔
8			✔	✔	✔	✔	✔	✔	✔	✔
9		✔	✔	✔	✔	✔	✔	✔	✔	✔

$$\begin{array}{r} 861 \\ +\ 472 \\ \hline \end{array}$$

Single-digit sums are the building blocks of multi-digit sums. For example, the addition problem shown involves the single-digit sums 1 + 2, 6 + 7, and 8 + 4. Single-digit sums also allow subtracting with the related differences, because of the relationship between addition and subtraction ($C - A$ is the unknown addend in $A + \square = C$). For example, to subtract 13 - 8 = ?, it is enough if seeing the numbers 13 and 8 prompts recall of the fact that 8 + 5 = 13; then the difference 13 - 8 must equal 5. Finally, knowing single-digit sums is also valuable for mental calculation with multi-digit numbers, as in the following examples:

$$37 + 4 = 30 + (7 + 4) = 30 + 11 = 41$$

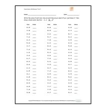
$$4 \times 8 = 2 \times (2 \times 8) = 2 \times 16 = 16 + 16 = (10 + 10) + (6 + 6) = 20 + 12 = 32.$$

Remembering single-digit sums and being fluent with related differences is an important goal (CCSS 2.OA.B.2). This goal needs to be reached by an intellectually valid, emotionally supportive learning path. The stages of that path are articulated in the *Progression* document[†], pp. 14–27.[‡]

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: Level 2 and 3 strategies as described in the *Progression* document under the heading "Using Level 2 and Level 3 strategies to extend addition and subtraction problem solving beyond 10, to problems within 20" (see pp. 14–17).

2:5 Write the value of each sum. Use as much time as you need. If you "just knew it," then draw a check mark, like this: 2 + 2 4✓



Click here for student handout 2:5

Answer

Click here for an answer key. Individual students' check marks may vary.

Click here for a student-facing version of the task.

Refer to the Standards

2.OA.B.2; MP.1, MP.6. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

- The task is designed to be worked on after all the sums in the addition table have been understood and practiced.
- The instructions say, "Use as much time as you need." Reasons for this include: (1) Differentiating between students on the basis of their speed isn't the purpose of the task. (2) More generally, speed isn't an important disciplinary value in mathematics. (3) Emphasizing speed in the mathematical community of the classroom can have negative effects on students' mathematics identity.

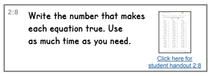
Extending the task

How might students drive the conversation further?

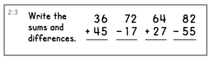
- Students could notice that a certain addend is absent from the student handout, or be asked which addend is absent. What might be the reason why no sums with this addend were included?
- Students could circle several addends they feel they could use more practice with.

Related Math Milestones tasks

2:8

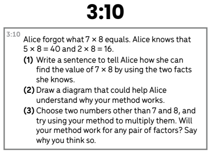


2:3

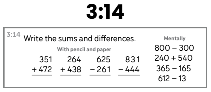


Task **2:8 Fluency within the Addition Table** includes problems about fact families, such as 12 - 5 = ☐, ☐ + 9 = 14, ☐ - 6 = 7, and 10 - ☐ = 5, in which an unknown number is sought that makes an equation true. Single-digit sums and related differences are involved in many calculations in grade 2 tasks, including **2:3 Fluency within 100 (Add/Subtract)** as well as the word problems (see the [Map of Addition and Subtraction Situations in K-2 Math Milestones]).

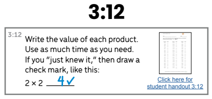
3:10



3:14

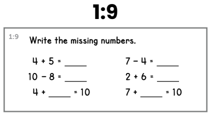


3:12

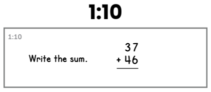


In later grades, fluency with sums and differences plays a role in multiplication strategies; see task **3:10 Alice's Multiplication Fact**. Fluency within the addition table is also a component in multi-digit addition and subtraction; see task **3:14 Fluency within 1000 (Add/Subtract)**. Task **3:12 Products of Single-Digit Numbers** is the analog of task 2:8 for multiplication and division.

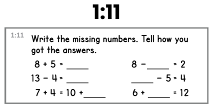
1:9



1:10



1:11



In earlier grades, tasks **1:9 Fluency within Ten**, **1:10 Two-Digit Addition**, and **1:11 Using Properties and Relationships** focus on addition and subtraction within 20.

Additional notes on the design of the task (continued)

- The instructions say, "If you 'just knew it,' then draw a check mark." This is intended to provide information about which single-digit sums are known from memory.
- The task includes only two sums that involve 1 as an addend, and it includes no sums that involve 0 as an addend. All such sums are instances of the general patterns 1 + n = n + 1 = the next number in the count sequence and 0 + n = n + 0 = n. (See "Extending the task.")

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:5? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 2:5? In what specific ways do they differ from 2:5?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:5 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2011, May 29). *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ For further discussion of such an interwoven process, see "Fluency Development Within and Across the Grades in IM K–5 Math™, part 1: Addition and Subtraction" (blog post by Caban and Aminata, 2021).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:6 Cutting a Rope

Teacher Notes

Math Milestones™

Central math concepts

Based on interviews with middle-grades students,† education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

In particular, task 2:6 involves addition and subtraction and the relationship between them. The mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between addition and subtraction can play

> 2:6 A rope is 32 feet long. The rope is cut into two pieces. One piece is 3 feet long. How long is the other piece?
>
> Equation model: ______________
>
> Answer: _____ feet

Answer

3 + ? = 32, 32 - 3 = ?, or another equivalent equation, or an equation showing the value of the difference or unknown addend, such as 3 + 29 = 32, 32 - 3 = 29, or another equivalent equation. The other piece is 29 feet long.

Click here for a student-facing version of the task.

Refer to the Standards

2.MD.B.5, 2.MD.B; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

The calculation involved in the task, 32 - 3, is one of the simplest calculations in the Math Milestones tasks for grade 2.

out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:‡

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems.

In particular, the situation type in task 2:6 is called "Put Together/Take Apart with One Addend Unknown." It is a Put Together/Take Apart situation because cutting a rope 'takes it apart' into two pieces; and more specifically, the situation is "Put Together/Take Apart with One Addend Unknown" because the initially unknown quantity is the length of one of the two pieces of rope.

During the primary grades, students work with all situation types and all variations in the known and unknown quantities, with quantities given as whole numbers. In the upper-elementary grades, these understandings of addition and subtraction are extended into working with fractional quantities. Although the algorithms for performing calculations with fractions are different from those for performing base-ten calculations with whole numbers, the underlying meanings and uses of addition and subtraction are the same regardless of whether the numbers involved are whole numbers, fractions, decimals, or even variables.

The context in task 2:6 involves length measurement. Thus, the "objects" being counted in this problem are length units. This contrasts with problems involving more tangible objects such as apples or grapes. It is important to include length contexts in grade 2 word problems, because relating addition and subtraction to length, including by working with number lines, is part of early preparation for working with fractional quantities in grades 3 and later.

Task 2:6 asks not only for the final answer but also for an equation model. An equation model is requested because compared to the answer alone ("29 feet"), an equation model is better evidence that students have comprehended the situation and its quantitative relationships. The equation model records the situation's mathematical structure so that students can discuss and reflect on it. Equivalent forms of the equation can also show the relationship between addition and subtraction.

It may happen that not all students write the same equation. That is an opportunity to discuss correspondences between equations and relationships to the situation described in the problem. The *Progression* document (p. 13) stresses that if textbooks and teachers model representations or solution methods, then "these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context."

Some equation models describe a situation in an algebraic way, such as ? + 3 = 32 (these are called *situation equations*); students may then rewrite a situation equation in a form more conducive to determining the

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:6? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:6? In what specific ways do they differ from 2:6?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:6 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 2, p. 9 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

unknown number, such as ? = 32 - 3 (this is called a *solution equation*). As observed in the *Progression* document,

> Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. (p. 13)

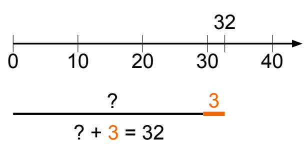
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: concepts of length measurement and length units; adding or subtracting a single-digit number to a two-digit number; working with the count sequence within 100; and writing and discussing situation equations and/or solution equations.

Extending the task

How might students drive the conversation further?

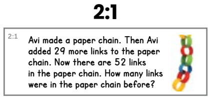
- For each number in an equation model for the situation, students could identify the quantity in the situation to which the number refers. For example, 32 refers to "the length of the rope before it was cut" or "the length of the two pieces if you put them together." Note that naming a quantity is different from naming the numerical value of the quantity.
- Students could create a schematic number line diagram representation of the situation and draw connections to the situation equation (see figure).



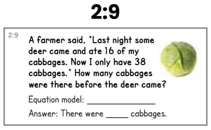
- Instead of cutting off a 3-foot piece of the rope, suppose we had cut the rope in half. How long would the pieces be in that case?

Related Math Milestones tasks

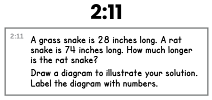
2:1



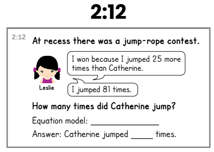
2:9



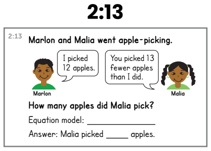
2:11



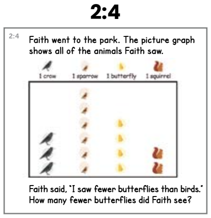
2:12



2:13



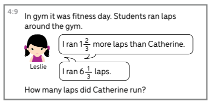
2:4



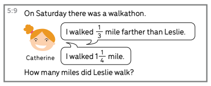
Other word problems and their situation types in grade 2 are as follows: tasks **2:1 Paper Chain**, *Add To with Start Unknown*; **2:9 Disappearing**

Cabbages, *Take From with Start Unknown*; **2:11 Grass Snake vs. Rat Snake**, *Compare with Difference Unknown ('how many more'/'how much longer' language*), in a context involving length; **2:12 Jump-Rope Contest**, *Compare with Smaller Quantity Unknown ('more' language)*; **2:13 Apple-Picking**, *Compare with Bigger Quantity Unknown ('fewer' language)*; and **2:4 Animals in the Park**, combination of *Put Together/Take Apart with Total Unknown* and *Compare with Difference Unknown ('how many fewer' language)*.

4:9



5:9



In later grades, tasks **4:9 Fitness Day** and **5:9 Walkathon** are addition/subtraction word problems of situation type *Compare with Smaller Quantity Unknown*.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones].

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

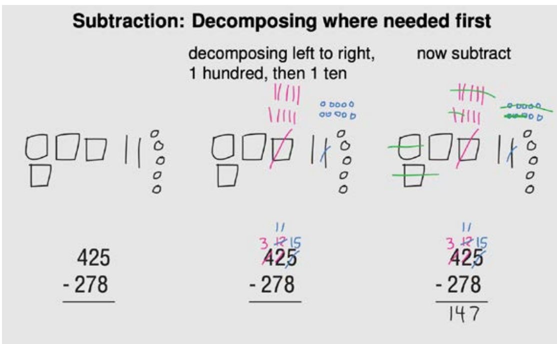
- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:7 Subtraction Regrouping

Teacher Notes

Central math concepts

The *Progression* document[†] (p. 10) illustrates a student-friendly approach to subtracting using an example of a problem that involves two decompositions (see figure). In this method, all of the necessary decomposing is done first, and then the subtractions are carried out at each place. (The figure also includes student drawings, which are not part of the method itself but rather show the meaning of the individual decomposition steps.)



All necessary decomposing is done first, then the subtractions are carried out. This highlights the two major steps involved and can help to inhibit the common error of subtracting a smaller digit on the top from a larger digit. Decomposing and subtracting can start from the left (as shown) or the right.

The technique of place-value decomposition reduces calculating a multi-digit difference to a sequence of calculations at each place. Therefore, fluency in calculating multi-digit differences (CCSS 3.NBT.A.2, 4.NBT.B.4) rests on fluency within the addition table (CCSS 2.OA.B.2).

Decomposing as necessary at each place removes the difficulty of not having enough ones, tens, or hundreds to subtract from. Crucially, these decomposition steps conserve the value of the number being subtracted from. Thus, when students decompose place value units to calculate a difference, they maintain the value of the number being subtracted from, even as they transform a difficult problem into an easier one. Something similar happens in later grades when students replace the problem $\frac{1}{2} - \frac{1}{3}$ by the equivalent but easier problem $\frac{3}{6} - \frac{2}{6}$, or when students replace the problem 0.28 ÷ 0.04 by the equivalent but easier problem 28 ÷ 4. Transforming a problem to make it easier to solve is one of the most powerful mathematical practices of all.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value to hundreds; and subtracting within the addition table.

2:7

(1) Write the number that makes the statement true.
6 hundreds + 3 tens + 4 ones
= 5 hundreds + _____ tens + 4 ones.

(2) How do you know your statement is true?

(3) Look for connections between your statement and this subtraction problem. What connections can you see?

$$\begin{array}{r} \overset{5\ 13}{\cancel{6}\cancel{3}4} \\ -\,482 \\ \hline 152 \end{array}$$

Answer

(1) 13. **(2)** Answers will vary, but reasoning should be based on the idea that 6 hundreds equals 5 hundreds and 10 tens; for example, 6 hundreds is the same as 5 hundreds + 10 tens, so 6 hundreds + 3 tens + 4 ones is the same as 5 hundreds + 10 tens + 3 tens + 4 ones, leading to the answer for part (1). **(3)** Answers will vary. Possible connections between the statement and the subtraction problem could include that the number being subtracted from is 634, which is 6 hundreds + 3 tens + 4 ones; and that according to the strikeouts and small numerals, one of the hundreds in 634 has been decomposed into 10 tens, making 13 tens, as in the statement. Answers to parts (2) and (3) could be supported with drawings or manipulatives.

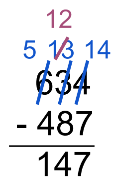
Click here for a student-facing version of the task.

Refer to the Standards

2.NBT.B.7, 2.NBT.B; MP.1, MP.2, MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Extending the task

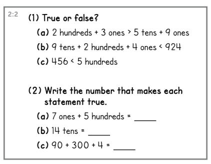
How might students drive the conversation further?



- Students could look for connections between the problem shown in the figure and the statement that 6 hundreds + 3 tens + 4 ones = 5 hundreds + 12 tens + 14 ones.
- Students could brainstorm the most common careless errors ("bugs") a student might make when subtracting numbers. Then students could formulate a tip for avoiding or finding and correcting these errors.

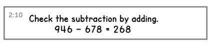
Related Math Milestones tasks

2:2

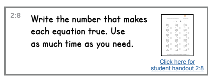

2:2 (1) True or false?
(a) 2 hundreds + 3 ones > 5 tens + 9 ones
(b) 9 tens + 2 hundreds + 4 ones < 924
(c) 456 < 5 hundreds

(2) Write the number that makes each statement true.
(a) 7 ones + 5 hundreds = _____
(b) 14 tens = _____
(c) 90 + 300 + 4 = _____

2:10

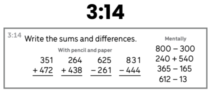

2:10 Check the subtraction by adding.
946 − 678 = 268

2:8

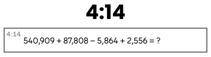

2:8 Write the number that makes each equation true. Use as much time as you need.
Click here for student handout 2:8

Task **2:2 Place Value to Hundreds** concentrates on place value concepts for three-digit numbers. Task **2:10 Three-Digit Addition/Subtraction** involves checking a three-digit difference by calculating a sum. Task **2:8 Fluency within the Addition Table** includes subtractions of the type that occur in each place when calculating a multi-digit difference.

3:14

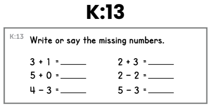

3:14 Write the sums and differences.
With pencil and paper
351 + 472
264 + 438
625 − 261
831 − 444
Mentally
800 − 300
240 + 540
365 − 165
612 − 13

4:14

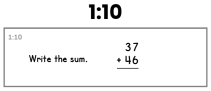

4:14 540,909 + 87,808 − 5,864 + 2,556 = ?

In later grades, task **3:14 Fluency within 1000 (Add/Subtract)** continues the multi-digit addition and subtraction progression to three-digit fluency, and task **4:14 Fluency with Multi-Digit Sums and Differences** marks the culmination of the progression in calculating multi-digit sums and differences.

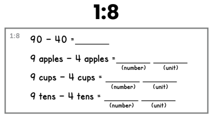
K:13


K:13 Write or say the missing numbers.
3 + 1 = _____ 2 + 3 = _____
5 + 0 = _____ 2 − 2 = _____
4 − 3 = _____ 5 − 3 = _____

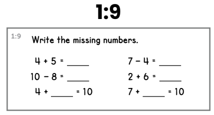
1:10


1:10 Write the sum.
37
+ 46

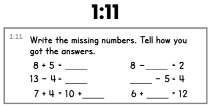
1:8


1:8 90 − 40 = _____
9 apples − 4 apples = _____ (number) _____ (unit)
9 cups − 4 cups = _____ (number) _____ (unit)
9 tens − 4 tens = _____ (number) _____ (unit)

1:9


1:9 Write the missing numbers.
4 + 5 = _____ 7 − 4 = _____
10 − 8 = _____ 2 + 6 = _____
4 + _____ = 10 7 + _____ = 10

1:11


1:11 Write the missing numbers. Tell how you got the answers.
8 + 5 = _____ 8 − _____ = 2
13 − 4 = _____ _____ − 5 = 4
7 + 4 = 10 + _____ 6 + _____ = 12

In earlier grades, task **K:13 Fluency within Five** includes subtractions with units of ones. Task **1:10 Two-Digit Addition** is a procedural task involving a sum of two two-digit numbers. Task **1:8 Subtracting Units** portrays a two-digit subtraction problem as a matter of subtracting two single-digit numbers of tens units. Sums of single-digit numbers and related differences are involved in tasks **1:9 Fluency within Ten** and **1:11 Using Properties and Relationships**.

Aspect(s) of rigor:

Concepts, Procedural skill and fluency

Additional notes on the design of the task

Connecting the steps of the subtraction procedure to the statement of equality is intended to promote sense-making about the procedure.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:7? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:7? In what specific ways do they differ from 2:7?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:7 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2015, March 6). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5, Number and Operations in Base Ten*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

2:7 Subtraction Regrouping

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:8 Fluency within the Addition Table

Teacher Notes

Central math concepts

Task 2:8 draws on memory, fluency, and conceptual understanding. In terms of conceptual understanding, the central mathematical idea in task 2:8 is that *C* - *A* is the unknown factor in *A* + ? = *C*. This idea expresses the mathematical relationship between addition and subtraction, whether for whole numbers, fractions, decimals, variables, variable expressions, or complex numbers. The 66 brief problems in task 2:8 involve different permutations of this relationship, as shown in the table.

Example Equation	Equation Type	How many are in task 2:8?
11 - 6 = ☐	Unknown Difference	34
☐ + 2 = 10 5 + ☐ = 12	Unknown Addend	18
☐ - 9 = 5	Unknown Total†	7
10 - ☐ = 6	Unknown Addend	7

Task 2:8 doesn't include equations of type Unknown Sum (for example, 3 + 8 = ☐), because sums like 3 + 8 are the topic of task **2:5 Sums of Single-Digit Numbers**. Whereas the problems in that task simply ask for the value of an expression like 3 + 8, the problems in task 2:8 ask for an unknown number that makes an equation true.

Remembering single-digit sums and being fluent with related differences is an important goal that supports a great deal of students' mathematical work in grades 3 and beyond. This goal needs to be reached by an intellectually valid, emotionally supportive learning path. The stages of that path are articulated in the *Progression* document,[‡] pp. 14–27.[§]

Relevant prior knowledge

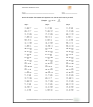
The following mathematics knowledge may be activated, extended, and deepened while students work on the task: the relationship between addition and subtraction; and remembered single-digit sums.

Extending the task

How might students drive the conversation further?

- Checking the equations with subtraction signs by adding can offer additional procedural practice and reinforce the relationship between multiplication and division (*C* - *A* is the unknown factor in *A* + ☐ = *C*).

2:8 Write the number that makes each equation true. Use as much time as you need.



Click here for student handout 2:8

Answer

Click here for an answer key.

Click here for a student-facing version of the task.

Refer to the Standards

2.OA.B.2; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency

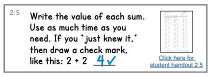
Additional notes on the design of the task

- The task is designed to be worked on after all the sums and related differences in the addition table have been understood and practiced.
- The instructions say, "Use as much time as you need." Reasons for this include: (1) Differentiating between students on the basis of their speed isn't the purpose of the task. (2) More generally, speed isn't an important disciplinary value in mathematics. (3) Emphasizing speed in the mathematical community of the classroom can have negative effects on students' mathematical identity.

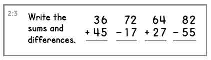
- Similarly, given a handful of cases of a completed equation (such as 15 - 7 = 8), students could write an equivalent equation (such as 8 + 7 = 15 or 15 - 8 = 7).

Related Math Milestones tasks

2:5


2:5 Write the value of each sum. Use as much time as you need. If you "just knew it," then draw a check mark, like this: 2 + 2 4✓
Click here for student handout 2:5

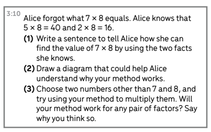
2:3


2:3 Write the sums and differences.

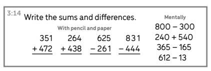
36	72	64	82
+ 45	− 17	+ 27	− 55

Task **2:5 Sums of Single-Digit Numbers** asks for the values of sums in the addition table. Single-digit sums and related differences are involved in many calculations in grade 2 tasks, including **2:3 Fluency within 100 (Add/Subtract)** as well as the word problems (see the [Map of Addition and Subtraction Situations in K–2 Math Milestones]).

3:10


3:10 Alice forgot what 7 × 8 equals. Alice knows that 5 × 8 = 40 and 2 × 8 = 16.
(1) Write a sentence to tell Alice how she can find the value of 7 × 8 by using the two facts she knows.
(2) Draw a diagram that could help Alice understand why your method works.
(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

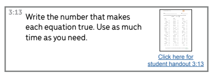
3:14


3:14 Write the sums and differences.

With pencil and paper: 351 + 472; 264 + 438; 625 − 261; 831 − 444

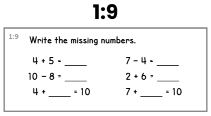
Mentally: 800 − 300; 240 + 540; 365 − 165; 612 − 13

3:13

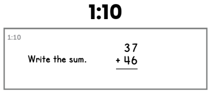

3:13 Write the number that makes each equation true. Use as much time as you need.
Click here for student handout 3:13

In later grades, fluency with sums and differences plays a role in multiplication strategies; see task **3:10 Alice's Multiplication Fact**. Fluency within the addition table is also a component in multi-digit addition and subtraction; see task **3:14 Fluency within 1000 (Add/Subtract)**. Task **3:13 Fluency within the Multiplication Table** is the analog of task 2:8 for multiplication and division.

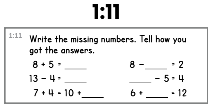
1:9


1:9 Write the missing numbers.
4 + 5 = ___ 7 − 4 = ___
10 − 8 = ___ 2 + 6 = ___
4 + ___ = 10 7 + ___ = 10

1:10


1:10 Write the sum. 37 + 46

1:11


1:11 Write the missing numbers. Tell how you got the answers.
8 + 5 = ___ 8 − ___ = 2
13 − 4 = ___ ___ − 5 = 4
7 + 4 = 10 + ___ 6 + ___ = 12

In earlier grades, tasks **1:9 Fluency within Ten**, **1:10 Two-Digit Addition**, and **1:11 Using Properties and Relationships** focus on addition and subtraction within 20.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:8? Locate 2–3 similar tasks in that unit. How are the tasks similar to each other, and to 2:8? In what specific ways do they differ from 2:8?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:8 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† From the *Progression*, p. 8: "Formal vocabulary for subtraction ('minuend' and 'subtrahend') is not needed for Kindergarten, Grade 1, and Grade 2, and may inhibit students seeing and discussing relationships between addition and subtraction. At these grades, the terms 'total' and 'addend' are sufficient for classroom discussion."

‡ Common Core Standards Writing Team. (2011, May 29). *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

§ For further discussion of such an interwoven process, see "Fluency Development Within and Across the Grades in IM K–5 Math™, part 1: Addition and Subtraction" (blog post by Caban and Aminata, 2021).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:9 Disappearing Cabbages

Teacher Notes

Central math concepts

Based on interviews with middle-grades students,[†] education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

In particular, task 2:9 involves addition and subtraction and the relationship between them. The mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between addition and subtraction can play

2:9 A farmer said, "Last night some deer came and ate 16 of my cabbages. Now I only have 38 cabbages." How many cabbages were there before the deer came?

Equation model: ____________

Answer: There were _____ cabbages.

Answer

? - 16 = 38, ? = 38 + 16, or another equivalent equation. There were 54 cabbages.

Click here for a student-facing version of the task.

Refer to the Standards

2.OA.A.1, 2.NBT.B.5; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application, Procedural skill and fluency

Additional notes on the design of the task

- The task presents the given information in the form of a monologue by the farmer. This could invite an approach of having students convey the task to each other by reciting the monologue.
- The calculation involved in the task, 38 + 16, is a two-digit addition problem which belongs to the fluency target in grade 2; see Teacher Notes for task **2:3 Fluency within 100 (Add/Subtract)**.

out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:[‡]

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems.

In particular, the situation type in task 2:9 is called "Take From with Start Unknown." It is a Take From situation because the deer took cabbages from the farmer; and more specifically, the situation is "Take From with Start Unknown" because the initially unknown quantity is how many cabbages the farmer had to begin with.

During the primary grades, students work with all situation types and all variations in the known and unknown quantities, with quantities given as whole numbers. In the upper-elementary grades, these understandings of addition and subtraction are extended into working with fractional quantities. Although the algorithms for performing calculations with fractions are different from those for performing base-ten calculations with whole numbers, the underlying meanings and uses of addition and subtraction are the same regardless of whether the numbers involved are whole numbers, fractions, decimals, or even variables.

Task 2:9 asks not only for the final answer but also for an equation model. An equation model is requested because compared to the answer alone ("54 cabbages"), an equation model is better evidence that students have comprehended the situation and its quantitative relationships. The equation model records the situation's mathematical structure so that students can discuss and reflect on it. Equivalent forms of the equation can also show the relationship between addition and subtraction.

It may happen that not all students write the same equation. That is an opportunity to discuss correspondences between equations and relationships to the situation described in the problem. The *Progression* document (p. 13) stresses that if textbooks and teachers model representations or solution methods, then "these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context."

Some equation models describe a situation in an algebraic way, such as ? - 16 = 38 (these are called *situation equations*); students may then rewrite a situation equation in a form more conducive to determining the unknown number, such as ? = 38 + 16 (this is called a *solution equation*). As observed in the *Progression* document,

> Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:9? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:9? In what specific ways do they differ from 2:9?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:9 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 2, p. 9 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. (p. 13)

Word problems vary considerably in the uses to which they put addition and subtraction, and they also vary in the complexity of the calculation required to obtain a final numerical answer. The overall challenge of a word problem depends on both the situational complexity and the computational complexity. The calculation in task 2:9 involves calculating a sum of two two-digit numbers in a case that also involves regrouping. Students work toward fluency with these calculations throughout the year. Students for whom the calculation 38 + 16 is time-consuming and/or effortful may need to be redirected to the context after obtaining the result 38 + 16 = 54, so as to relate the numbers in this equation to the context and answer the question in the task.

The sum 38 + 16 could be calculated in many ways using place value, properties of operations, and/or the relationship between addition and subtraction. A student could use a pencil and paper method based on tens and ones (see Teacher Notes for task **2:3 Fluency within 100 (Add/Subtract)**). A student could calculate mentally in ways such as in the following examples:

- 38 + 10 = 48, 48 + 2 = 50, 50 + 4 = 54.
- 38 + 16 = 40 + 14 = 54.

Such mental calculation methods should be discussed and compared in connection with corresponding written equations.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: adding and/or subtracting with two-digit numbers; working with the count sequence within 100; and writing and discussing situation equations and/or solution equations.

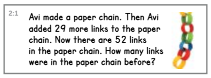
Extending the task

How might students drive the conversation further?

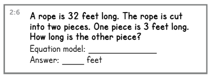
- For each number in an equation model for the situation, students could identify the quantity in the situation to which the number refers. For example, 38 refers to “the number of cabbages after the deer came.” Note that naming a quantity is different from naming the numerical value of the quantity.
- Students could imagine the deer coming back the next night and eating 16 more cabbages. How many cabbages would be left after that?

Related Math Milestones tasks

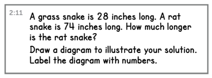
2:1



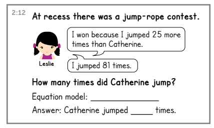
2:6



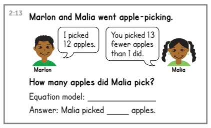
2:11



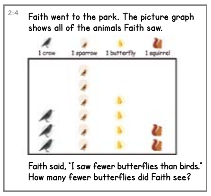
2:12



2:13

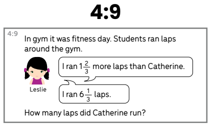


2:4

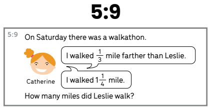


Other word problems and their situation types in grade 2 are as follows: tasks **2:1 Paper Chain**, *Add To with Start Unknown*; **2:6 Cutting a Rope**, *Put Together with One Addend Unknown*, in a context involving length; **2:11 Grass Snake vs. Rat Snake**, *Compare with Difference Unknown ('how many more'/'how much longer' language)*, in a context involving length; **2:12 Jump-Rope Contest**, *Compare with Smaller Quantity Unknown ('more' language)*; **2:13 Apple-Picking**, *Compare with Bigger Quantity Unknown ('fewer' language)*; and **2:4 Animals in the Park**, combination of *Put Together/Take Apart with Total Unknown* and *Compare with Difference Unknown ('how many fewer' language)*.

4:9



5:9



In later grades, tasks **4:9 Fitness Day** and **5:9 Walkathon** are addition/subtraction word problems of situation type *Compare with Smaller Quantity Unknown*.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones].

2:9 Disappearing Cabbages

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

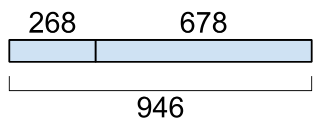
- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:10 Three-Digit Addition/Subtraction

Teacher Notes

Central math concepts

To know how to check the subtraction in an equation like 946 - 678 = 268, students must understand the relationships between the terms in the equation. The diagram shows one way to represent the three quantities and their relationship. This diagram could be used to generate several equations, forming a “fact family”:



268 + 678 = 946

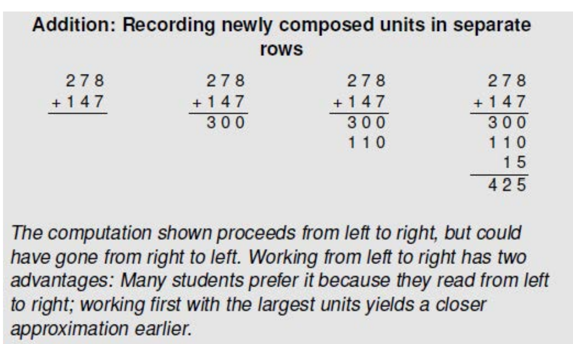
678 + 268 = 946

946 - 678 = 268

946 - 268 = 678

In general, the mathematical relationship between addition and subtraction is that *C* - *A* is the unknown addend in *A* + ? = *C*. Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

To calculate sums such as 678 + 268, students in grade 2 learn to use an efficient, accurate, and generalizable strategy that can develop further to meet the procedural expectations in grade 3 and grade 4. For example, pages 9 and 10 of the *Progression* document† show two calculation methods for three-digit addition. Images of these methods are presented below, along with the two points of commentary from the *Progression* document.



Addition: Recording newly composed units in separate rows

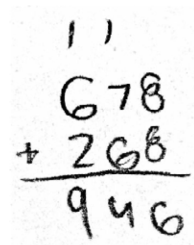
```
 278      278      278      278
+147     +147     +147     +147
         300      300      300
                  110      110
                            15
                           425
```

The computation shown proceeds from left to right, but could have gone from right to left. Working from left to right has two advantages: Many students prefer it because they read from left to right; working first with the largest units yields a closer approximation earlier.

2:10 Check the subtraction by adding.
946 − 678 = 268

Answer

See the example. Students might use different strategies and/ or algorithms than the one shown in the example.



Click here for a student-facing version of the task.

Refer to the Standards

2.NBT.B.7; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Procedural skill and fluency

Additional notes on the design of the task

The sum 678 + 268 involves two steps of composing a new unit.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:10? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 2:10? In what specific ways do they differ from 2:10?

Addition: Recording newly composed units in the same row

Add the ones, 8 + 7, and record these 15 ones with 1 on the line in the tens column and 5 below in the ones place.	Add the tens, 7 + 4 + 1, and record these 12 tens with 1 on the line in the hundreds column and 2 below in the tens place.	Add the hundreds, 2 + 1 + 1, and record these 4 hundreds below in the hundreds column.

Digits representing newly composed units are placed below the addends, on the line. This placement has several advantages. Each two-digit partial sum (e.g., "15") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 5) rather than "write the 5 and carry the 1" (write 5, then 1).

- "The first written method is a helping step variation that generalizes to all numbers in base ten but becomes impractical because of writing so many zeros. Students can move from this method to the second method (or another compact method) by seeing how the steps of the two methods are related. Some students might make this transition in Grade 2, some in Grade 3, but all need to make it by Grade 4 where fluency requires a more compact method."
- "Counting-on and adding-on methods become even more difficult with numbers over 1000. If they arise from students, they should be discussed. But the major focus for addition within 1000 needs to be on methods such as those [shown in the two images] that are simpler for students and lead toward fluency (e.g., [as in the first image]) or are sufficient for fluency (e.g., [as in the second image])."

The *Progression* document discusses the ways students can make sense of these methods in relation to concepts of place value. Also implicit in performing such calculations is learning the counting sequence from 100 to 1,000; see the *Progression* document, p. 8.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value concepts; single-digit sums; regrouping in addition; and reasoning about addends, sums, and differences.

Extending the task

How might students drive the conversation further?

- Students could use a written subtraction method to verify directly that 946 - 678 = 268.
- Students could look for correspondences in a piece of written work for 678 + 268 = 946 and a piece of written work for 946 - 678 = 268. (Decomposition steps in the subtraction calculation can be connected to composition steps in the addition calculation.)

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:10 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2015, March 6). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5, Number and Operations in Base Ten*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

- Students could be given a true equation, something impressive like 82,734 - 23,789 = 58,945; assuming this is true, what must be the result of subtracting 82,734 - 58,945 = ? (Check using a calculator to see if the reasoning worked!)

Related Math Milestones tasks

2:2

2:2 (1) True or false?
(a) 2 hundreds + 3 ones > 5 tens + 9 ones
(b) 9 tens + 2 hundreds + 4 ones < 924
(c) 456 < 5 hundreds

(2) Write the number that makes each statement true.
(a) 7 ones + 5 hundreds = ____
(b) 14 tens = ____
(c) 90 + 300 + 4 = ____

2:5

2:5 Write the value of each sum. Use as much time as you need. If you "just knew it," then draw a check mark, like this: 2 + 2 4✓
Click here for student handout 2:5

2:8

2:8 Write the number that makes each equation true. Use as much time as you need.
Click here for student handout 2:8

2:3

2:3 Write the sums and differences.
36 + 45, 72 − 17, 64 + 27, 82 − 55

The place value ideas necessary for calculating three-digit sums and differences are the subject of task **2:2 Place Value to Hundreds**. Tasks **2:5 Sums of Single-Digit Numbers** and **2:8 Fluency within the Addition Table** involve the single-digit sums and related differences upon which multi-digit addition and subtraction algorithms are built. Two-digit sums and differences are addressed from a fluency perspective in task **2:3 Fluency within 100 (Add/Subtract)**.

3:14

3:14 Write the sums and differences.
With pencil and paper: 351 + 472, 264 + 438, 625 − 261, 831 − 444
Mentally: 800 – 300, 240 + 540, 365 – 165, 612 – 13

4:14

4:14 540,909 + 87,808 – 5,864 + 2,556 = ?

In later grades, task **3:14 Fluency within 1000 (Add/Subtract)** involves three-digit sums and differences, some chosen for pencil-and-paper calculation, others chosen for mental calculation. Task **4:14 Fluency with Multi-Digit Sums and Differences** marks the culmination of the progression for multi-digit addition and subtraction.

1:10

1:10 Write the sum. 37 + 46

1:8

1:8 90 − 40 = ____
9 apples − 4 apples = ____ (number) ____ (unit)
9 cups − 4 cups = ____ (number) ____ (unit)
9 tens − 4 tens = ____ (number) ____ (unit)

1:9

1:9 Write the missing numbers.
4 + 5 = ____ 7 − 4 = ____
10 − 8 = ____ 2 + 6 = ____
4 + ____ = 10 7 + ____ = 10

1:11

1:11 Write the missing numbers. Tell how you got the answers.
8 + 5 = ____ 8 − ____ = 2
13 − 4 = ____ ____ − 5 = 4
7 + 4 = 10 + ____ 6 + ____ = 12

In earlier grades, task **1:10 Two-Digit Addition** is a procedural task involving a sum of two two-digit numbers. Task **1:8 Subtracting Units** portrays a two-digit subtraction problem as a matter of subtracting two single-digit numbers of tens units. Sums of single-digit numbers and related differences are involved in tasks **1:9 Fluency within Ten** and **1:11 Using Properties and Relationships**.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:11 Grass Snake vs. Rat Snake

Teacher Notes

Central math concepts

Based on interviews with middle-grades students,† education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

For addition and subtraction specifically, the mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. (One might paraphrase this statement by saying that, "Given a total and one part, subtraction finds the other part.") Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the

2:11 A grass snake is 28 inches long. A rat snake is 74 inches long. How much longer is the rat snake?

Draw a diagram to illustrate your solution. Label the diagram with numbers.

Answer

The rat snake is 46 inches longer than the grass snake. Diagrams may vary but might take the form of a tape diagram (appropriately labeled with the values 28 and 74 and the length difference value 46); a number line diagram (with locations indicated for the numbers 0, 28, and 74 and with the interval between 28 and 74 labeled with 46); or a more or less abstract rendering of the snakes themselves with appropriate labeling of lengths and the length difference.

Click here for a student-facing version of the task.

Refer to the Standards

2.MD.B.5, 2.MD.B; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application, Procedural skill and fluency

Additional notes on the design of the task

- The task does not include an illustration, because creating a diagram is part of the task.

mathematical relationship between addition and subtraction can play out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems. In particular, the situation type in task 2:11 is called "Compare with Difference Unknown."[‡]

As noted in the *Progression* document, "One reason Compare problems are more advanced than the other two major types is that in Compare problems, one of the quantities (the difference) is not present in the situation physically, and must be conceptualized and constructed in a representation, by showing the 'extra' that when added to the smaller unknown makes the total equal to the bigger unknown or by finding this quantity embedded within the bigger unknown" (p. 12).

The context in task 2:11 sets up a comparison of two length measurements. Thus, the "objects" being counted in this problem are length units. This situation contrasts with Compare problems involving more tangible objects such as apples or grapes. It is important to include length contexts in grade 2 word problems, because relating addition and subtraction to length, including by working with number lines, is part of early preparation for working with fractional quantities in grades 3 and later.

Task 2:11 asks not only for the final answer but also for a diagram. A diagram is requested because compared to the answer alone ("46 inches"), a diagram is better evidence that students have comprehended the situation and its quantitative relationships. The diagram records the situation's mathematical structure so that students can discuss and reflect on it. The diagram can also illustrate the relationship between addition and subtraction.

Not all students will create the same diagram to illustrate their solution. The *Progression* document (p. 13) stresses that "If textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context."

Word problems vary considerably in the uses to which they put addition and subtraction, and they also vary in the complexity of the calculation required to obtain a final numerical answer. The overall challenge of a word problem depends on both the situational complexity and the computational complexity. The calculation in task 2:11 involves calculating a difference of two two-digit numbers in a case that also involves regrouping. Calculations of this kind are among the grade 2 fluency targets, but they are also new to students in this grade. Students for whom the calculation 74 - 28 is time-consuming and/or effortful may need to be redirected to the context after obtaining the result 74 - 28 = 46, so

Additional notes on the design of the task (continued)

- Grass snake and rat snake are real snake species, and the two individuals described in the task have reasonable lengths for their species. Students may not have heard of either species, however. After working on the task, some students might be interested to discuss the two kinds of snake.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:11? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:11? In what specific ways do they differ from 2:11?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:11 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ For the other situation types, see Table 2, p. 9 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona). The *Progression* document has extensive discussion of aspects of teaching and learning about Compare problems. Note that Compare problems are studied in Grades 1 and 2; quotations from the *Progression* document are taken from the section for Grade 1, because Compare problems are discussed there first.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

as to relate the numbers in this equation to the context and answer the question in the task.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: calculating sums of two two-digit numbers; writing equations to describe quantitative relationships; and fundamental concepts of multiplication with whole numbers.

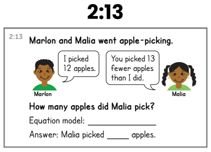
Extending the task

How might students drive the conversation further?

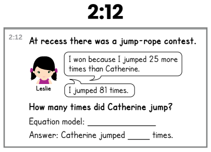
- Students with an interest in the topic could compare the lengths of the snakes in the task to typical lengths of some other snake species they know about, such as a cobra, anaconda, or even a prehistoric snake such as Titanoboa or Sanajeh. (The lengths of the snakes could be provided in inches.)

Related Math Milestones tasks

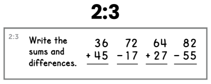
2:13



2:12

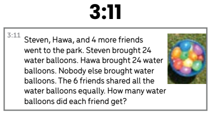


2:3

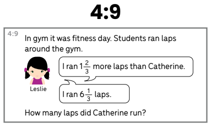


Task **2:13 Apple-Picking** is a word problem of situation type *Compare with Bigger Unknown*. Task **2:12 Jump-Rope Contest** is a word problem of situation type *Compare with Smaller Unknown*. Task **2:3 Fluency within 100 (Add/Subtract)** is a fluency task involving two-digit sums and differences.

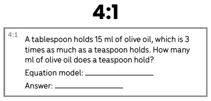
3:11



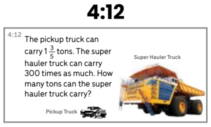
4:9



4:1

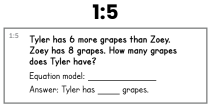


4:12

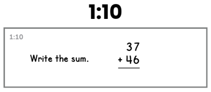


In later grades, task **3:11 Water Balloons** is a multi-step word problem involving more than one operation. Task **4:9 Fitness Day** is a word problem of situation type *Compare with Smaller Unknown* that involves fraction quantities. Tasks **4:1 A Tablespoon of Oil** and **4:12 Super Hauler Truck** are also comparison problems, but the comparisons in these tasks are multiplicative, not additive.

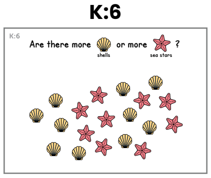
1:5



1:10



K:6



In earlier grades, task **1:5 Tyler's Grapes** is a word problem of situation type *Compare with Smaller Unknown*. Task **1:10 Two-Digit Addition** involves finding the sum of two two-digit numbers. Task **K:6 More Shells or More Stars?** involves comparison but without finding how many more.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:12 Jump-Rope Contest

Teacher Notes

Central math concepts

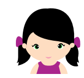
Based on interviews with middle-grades students,† education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

In particular, task 2:12 involves addition and subtraction and the relationship between them. The mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. (One might paraphrase this statement by saying that, "Given a total and one part, subtraction finds the other part.") Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

2:12 At recess there was a jump-rope contest.



Leslie

I won because I jumped 25 more times than Catherine.

I jumped 81 times.

How many times did Catherine jump?

Equation model: ______________

Answer: Catherine jumped _____ times.

Answer

? + 25 = 81, 81 - 25 = ?, or another equivalent equation. Catherine jumped 56 times.

Click here for a student-facing version of the task.

Refer to the Standards

2.OA.A.1, 2.NBT.B.5; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application, Procedural skill and fluency

Additional notes on the design of the task

- Word problems involving Compare situations can sometimes consist of complex text. Therefore, task 2:12 presents the given information in the form of a monologue by Leslie. This could also invite an approach of having students convey the task to each other by reciting the monologue.

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between addition and subtraction can play out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:‡

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems.

In particular, the situation type in task 2:12 is called "Compare with Smaller Unknown." It is a Compare situation because Leslie is using subtraction to compare her number of jumps with Catherine's number of jumps; and more specifically, the situation is "Compare with Smaller Unknown" because the initially unknown quantity is how many times Catherine jumped (and Catherine jumped fewer times).

During the primary grades, students work with all situation types and all variations in the known and unknown quantities, with quantities given as whole numbers. In the upper-elementary grades, these understandings of addition and subtraction are extended into working with fractional quantities. Although the algorithms for performing calculations with fractions are different from those for performing base-ten calculations with whole numbers, the underlying meanings and uses of addition and subtraction are the same regardless of whether the numbers involved are whole numbers, fractions, decimals, or even variables.

Task 2:12 asks not only for the final answer but also for an equation model. An equation model is requested because compared to the answer alone ("56 jumps"), an equation model is better evidence that students have comprehended the situation and its quantitative relationships. The equation model records the situation's mathematical structure so that students can discuss and reflect on it. Equivalent forms of the equation can also show the relationship between addition and subtraction.

It may happen that not all students write the same equation. That is an opportunity to discuss correspondences between equations and relationships to the situation described in the problem. The *Progression* document (p. 13) stresses that "If textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context."

Some equation models describe a situation in an algebraic way, such as ? + 25 = 81 (these are called *situation equations*); students may then rewrite a situation equation in a form more conducive to determining the unknown number, such as ? = 81 - 25 (this is called a *solution equation*). As observed in the *Progression* document,

Additional notes on the design of the task (continued)

- The calculation involved in the task, 81 - 25, is a two-digit subtraction problem which belongs to the fluency target in grade 2; see Teacher Notes for task **2:3 Fluency within 100 (Add/Subtract)**.
- Leslie and Catherine also appear in tasks **4:9 Fitness Day** and **5:9 Walkathon**.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:12? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:12? In what specific ways do they differ from 2:12?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:12 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report.* San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 2, p. 9 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

> Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. (p. 13)

Word problems vary considerably in the uses to which they put addition and subtraction, and they also vary in the complexity of the calculation required to obtain a final numerical answer. The overall challenge of a word problem depends on both the situational complexity and the computational complexity. The calculation in task 2:12 involves calculating a difference of two two-digit numbers in a case that also involves regrouping. Calculations of this kind are among the grade 2 fluency targets, but they are also new to students in this grade. Students for whom the calculation 81 - 25 is time-consuming and/or effortful may need to be redirected to the context after obtaining the result 81 - 25 = 56, so as to relate the numbers in this equation to the context and answer the question in the task.

The difference 81 - 25 could be calculated in many ways using place value, properties of operations, and/or the relationship between addition and subtraction. A student could use a pencil and paper method based on tens and ones (see Teacher Notes for task **2:3 Fluency within 100 (Add/Subtract)**). A student could calculate mentally in ways such as in the following examples:

- "81 - 21 = 60, take away 4 more is 56."
- "25 + **50** is 75, plus **6** more is 81, so 25 + **56** = 81."
- 81 - 25 = 71 - 15 = 61 - 5 = 60 - 5 + 1 = 55 + 1 = 56.

Such mental calculation methods should be discussed and compared in connection with corresponding written equations.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: adding and/or subtracting with two-digit numbers; working with the count sequence within 100; and writing and discussing situation equations and/or solution equations.

Extending the task

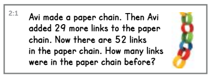
How might students drive the conversation further?

- For each number in an equation model for the situation, students could identify the quantity in the situation to which the number refers. For example, 81 refers to "the number of times Leslie jumped." Note that naming a quantity is different from naming the numerical value of the quantity.
- Students could add a third person to the story who outdoes both Leslie and Catherine. Students could create new dialogue and ask a new

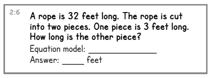
question. For example, new dialogue could say, “Oh yeah? I jumped 100 times,” and a new question could be, “How many more times did the new person jump than Leslie?”

Related Math Milestones tasks

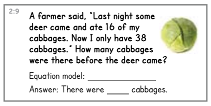
2:1



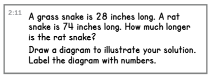
2:6



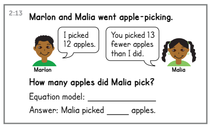
2:9



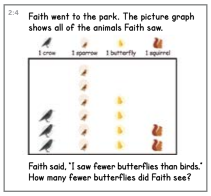
2:11



2:13

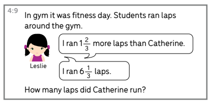


2:4

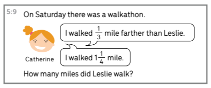


Other word problems and their situation types in grade 2 are as follows: tasks **2:1 Paper Chain**, *Add To with Start Unknown*; **2:6 Cutting a Rope**, *Put Together with One Addend Unknown*, in a context involving length; **2:9 Disappearing Cabbages**, *Take From with Start Unknown*; **2:11 Grass Snake vs. Rat Snake**, *Compare with Difference Unknown ('how many more'/'how much longer' language)*, in a context involving length; **2:13 Apple-Picking**, *Compare with Bigger Quantity Unknown ('fewer' language)*; and **2:4 Animals in the Park**, combination of *Put Together/Take Apart with Total Unknown* and *Compare with Difference Unknown ('how many fewer' language)*.

4:9



5:9



In later grades, tasks **4:9 Fitness Day** and **5:9 Walkathon** are addition/subtraction word problems of situation type Compare with Smaller Quantity Unknown.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones](#).

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:13 Apple-Picking

Teacher Notes

Central math concepts

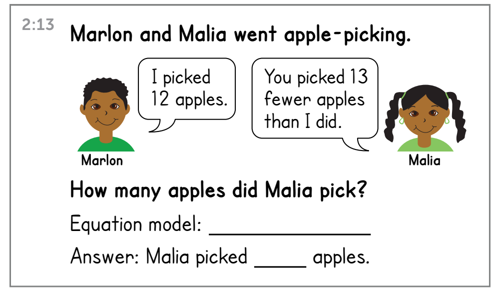
Based on interviews with middle-grades students,[†] education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract...; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "...if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

For addition and subtraction specifically, the mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. (One might paraphrase this statement by saying that, "Given a total and one part, subtraction finds the other part.") Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the



Answer

? - 13 = 12, 12 + 13 = ?, or another equivalent equation. Malia picked 25 apples.

Click here for a student-facing version of the task.

Refer to the Standards

2.OA.A.1; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

Word problems involving Compare situations can sometimes consist of complex text. Therefore, task 2:13 presents the given information in the form of a dialogue between Marlon and Malia. This could also invite an approach of having students convey the task to each other by acting out the dialogue.

mathematical relationship between addition and subtraction can play out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems. In particular, the situation type in task 2:13 is called "Compare with Bigger Unknown." It is a Compare situation because Malia is using subtraction to compare her 25 apples with Marlon's 12 apples; and more specifically, the situation is "Compare with Bigger Unknown" because the initially unknown quantity is how many apples Malia has (and Malia has the bigger apple haul).

For the other situation types, see Table 2, p. 9 of the *Progression* document.[‡] It has extensive discussion of aspects of teaching and learning about Compare problems,[§] including the following points (pp. 12, 13).

- "One reason Compare problems are more advanced than the other two major types is that in Compare problems, one of the quantities (the difference) is not present in the situation physically, and must be conceptualized and constructed in a representation, by showing the "extra" that when added to the smaller unknown makes the total equal to the bigger unknown or by finding this quantity embedded within the bigger unknown."
- "The language of comparisons is also difficult. For example, 'Julie has three more apples than Lucy' tells both that Julie has more apples and that the difference is three. Many students 'hear' the part of the sentence about who has more, but do not initially hear the part about how many more; they need experience hearing and saying a separate sentence for each of the two parts in order to comprehend and say the one-sentence form."
- "Another language issue is that the comparing sentence might be stated in either of two related ways, using more or less. Students need considerable experience with less to differentiate it from more; some children think that less means more."

Task 2:13 asks not only for the final answer but also for an equation model. An equation model is requested because compared to the answer alone ("25 apples"), an equation model is better evidence that students have comprehended the situation and its quantitative relationships. The equation model records the situation's mathematical structure so that students can discuss and reflect on it. Equivalent forms of the equation can also show the relationship between addition and subtraction.

It may happen that not all students write the same equation. That is an opportunity to discuss correspondences between equations and relationships to the situation described in the problem. The *Progression* document (p. 13) stresses that "[i]f textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:13? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:13? In what specific ways do they differ from 2:13?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:13 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report.* San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ Common Core Standards Writing Team. (2011, May 29). *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

§ Compare problems are studied in grades 1 and 2. Quotations from the *Progression* document are taken from the section for grade 1 because Compare problems are first discussed there.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context."

Some equation models describe a situation in an algebraic way, such as ? - 12 = 13 (these are called *situation equations*); students may then rewrite a situation equation in a form more conducive to determining the unknown number, such as ? = 12 + 13 (this is called a *solution equation*). As observed in the *Progression* document,

> Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. (p. 13)

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: calculating sums of two two-digit numbers; writing equations to describe quantitative relationships; and fundamental concepts of addition and subtraction.

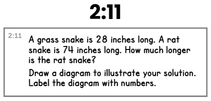
Extending the task

How might students drive the conversation further?

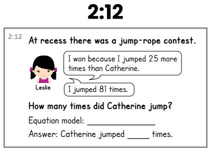
- For each number in an equation model for the situation, students could identify the quantity in the situation to which the number refers. For example, 25 refers to "the number of apples Malia picked." Note that naming a quantity is different from naming the numerical value of the quantity.
- Students could add a third person to the story who outdoes both Marlon and Malia. Students could create new dialogue and ask a new question. For example, new dialogue could say, "Oh yeah? I picked 4 more apples than Malia," and a new question could be, "How many more apples than Marlon did the new person pick?"

Related Math Milestones tasks

2:11

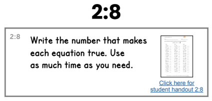

2:11 A grass snake is 28 inches long. A rat snake is 74 inches long. How much longer is the rat snake?
Draw a diagram to illustrate your solution. Label the diagram with numbers.

2:12


2:12 At recess there was a jump-rope contest.

How many times did Catherine jump?
Equation model: ______________
Answer: Catherine jumped _____ times.

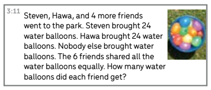
2:8


2:8 Write the number that makes each equation true. Use as much time as you need.
Click here for student handout 2:8

Task **2:11 Grass Snake vs. Rat Snake** is a word problem of situation type *Compare with Difference Unknown* that relates addition and subtraction to length. Task **2:12 Jump-Rope Contest** is a word problem of situation type *Compare with Smaller Unknown*. Task **2:8 Fluency within the**

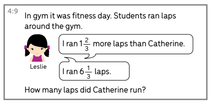
Addition Table is a fluency task involving the relationship between addition and subtraction. See also the [Map of Addition and Subtraction Situations in K–2 Math Milestones](#).

3:11



3:11 Steven, Hawa, and 4 more friends went to the park. Steven brought 24 water balloons. Hawa brought 24 water balloons. Nobody else brought water balloons. The 6 friends shared all the water balloons equally. How many water balloons did each friend get?

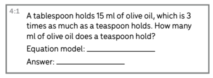
4:9



4:9 In gym it was fitness day. Students ran laps around the gym.

How many laps did Catherine run?

4:1



4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ______________

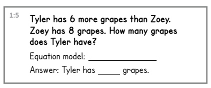
Answer: ______________

4:12

4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

In later grades, task **3:11 Water Balloons** is a multi-step word problem involving more than one operation. Task **4:9 Fitness Day** is a word problem of situation type *Compare with Smaller Unknown* that involves fraction quantities. Tasks **4:1 A Tablespoon of Oil** and **4:12 Super Hauler Truck** are also comparison problems, but the comparisons in these tasks are multiplicative, not additive.

1:5

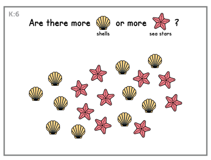


1:5 Tyler has 6 more grapes than Zoey. Zoey has 8 grapes. How many grapes does Tyler have?

Equation model: ______________

Answer: Tyler has _____ grapes.

K:6



In earlier grades, task **1:5 Tyler's Grapes** is a word problem of situation type *Compare with Smaller Unknown*. Task **K:6 More Shells or More Stars?** involves comparison but without finding how many more.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

2:14 Correcting a Shape Answer

Teacher Notes

Central math concepts

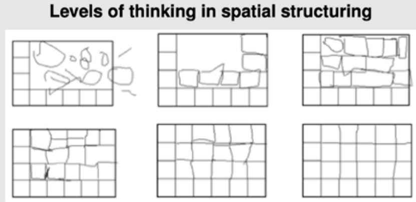
As explained in the *Progression* document (p. 2),† the three themes of elementary-grades geometry are:

- Spatial relations and spatial structuring;
- Composing and decomposing shapes; and
- Reasoning with shape components, shape properties, and shape categories.

These three themes are involved in the three parts (a), (b), and (c) of task 2:14.

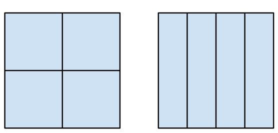
In part (a), a rectangle is spatially structured into an array of squares. Spatial structuring is "the mental operation of constructing an organization or form for an object or set of objects in space," and it builds on students' experiences with shape composition (p. 4). Such spatial structuring will be used in grade 3 to understand area measurement for rectangles, and in grade 5 to understand volume measurement for right rectangular prisms. Spatial structuring is also involved in partitioning of wholes during division and fraction reasoning in grades 3–6. Thus, "spatial structuring precedes meaningful mathematical use of the structures" (p. 4). The figure shows levels of spatial structuring portrayed by different students in response to an assessment task (p. 11).

Levels of thinking in spatial structuring



Levels of thinking portrayed by different students as they attempted to complete a drawing of an array of squares, given one column and row. This was an assessment, not an instructional task.

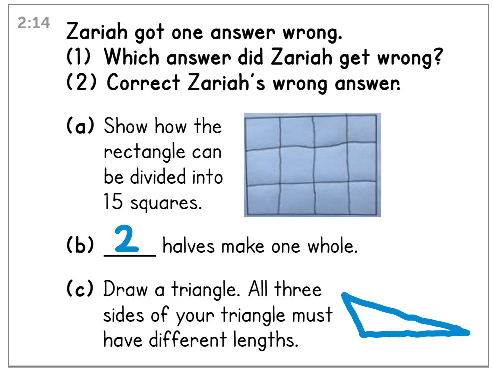
In part (b), two halves are composed to make one whole. This part of task 2:14 is linguistic in nature, but it draws on students' experiences with shape composition, as in the figure for example. Composing into a whole, and decomposing into halves and fourths, also prepares for fraction work in grade 3.



Decomposing a square into fourths in two different ways. A fourth on the left can be shown to be equal in size (equal in area) to a fourth on the right by decomposing it into parts and composing the parts into a different shape.

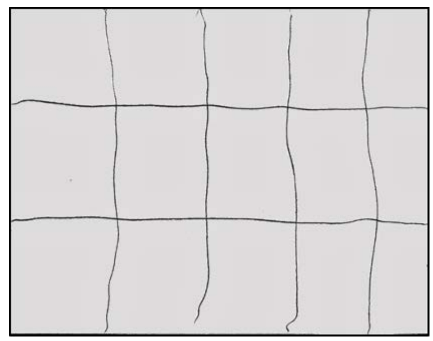
In part (c), a triangle is discussed in terms of its attributes. The *Progression* document (p. 3) describes three *levels of geometric thinking* that describe increasing sophistication with this learning progression:

- **Visual/Syncretic level**. Students recognize shapes, for example, a rectangle "looks like a door."
- **Descriptive level**. Students perceive properties of shapes, for example, a rectangle has four sides, all its sides are straight, opposite sides have equal length.



Answer

(1) Answer (a) is wrong. **(2)** See the example of a corrected answer for (a).



Click here for a student-facing version of the task.

Refer to the Standards

2.G.A; MP.1, MP.3, MP.5, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

- **Analytic level**. Students characterize shapes by their properties, for example, a rectangle has opposite sides of equal length and four right angles.
- **Abstract level**. Students understand, for example, that a rectangle is a parallelogram because it has all the properties of parallelograms.

Task 2:14 connects properties of shapes at the Analytic level with students' focus on length measurement in this grade.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: composing and decomposing squares and rectangles; identifying and naming shape attributes; and reasoning about length and/or measuring length.

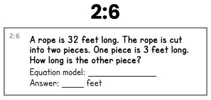
Extending the task

How might students drive the conversation further?

- Students could extend part (b) by drawing shapes that illustrate 2 halves making 1 whole, aiming for variety in the resulting drawings.
- Students could extend further by drawing shapes that illustrate 4 fourths making 1 whole, aiming for variety in the resulting drawings.

Related Math Milestones tasks

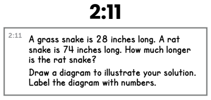
2:6


2:6 A rope is 32 feet long. The rope is cut into two pieces. One piece is 3 feet long. How long is the other piece?

Equation model: ______________

Answer: _____ feet

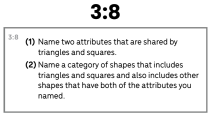
2:11


2:11 A grass snake is 28 inches long. A rat snake is 74 inches long. How much longer is the rat snake?

Draw a diagram to illustrate your solution. Label the diagram with numbers.

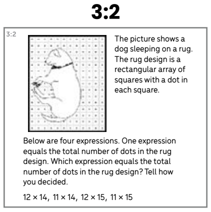
Spatial structuring and composing/decomposing are involved in length measurement (iterating length units), and composing and decomposing is involved in adding and subtracting lengths, as in tasks **2:6 Cutting a Rope** and **2:11 Grass Snake vs. Rat Snake**.

3:8


3:8 **(1)** Name two attributes that are shared by triangles and squares.

(2) Name a category of shapes that includes triangles and squares and also includes other shapes that have both of the attributes you named.

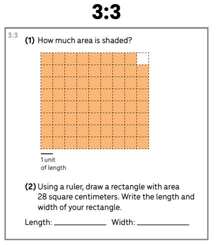
3:2


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

$12 \times 14,\ 11 \times 14,\ 12 \times 15,\ 11 \times 15$

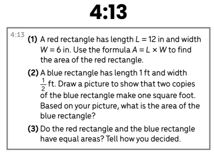
3:3


3:3 **(1)** How much area is shaded?

(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ___________ Width: ___________

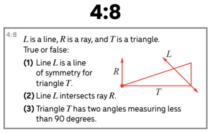
4:13


4:13 **(1)** A red rectangle has length $L = 12$ in and width $W = 6$ in. Use the formula $A = L \times W$ to find the area of the red rectangle.

(2) A blue rectangle has length 1 ft and width $\frac{1}{2}$ ft. Draw a picture to show that two copies of the blue rectangle make one square foot. Based on your picture, what is the area of the blue rectangle?

(3) Do the red rectangle and the blue rectangle have equal areas? Tell how you decided.

4:8

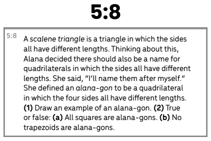

4:8 L is a line, R is a ray, and T is a triangle. True or false:

(1) Line L is a line of symmetry for triangle T.

(2) Line L intersects ray R.

(3) Triangle T has two angles measuring less than 90 degrees.

5:8


5:8 A *scalene triangle* is a triangle in which the sides all have different lengths. Thinking about this, Alana decided there should also be a name for quadrilaterals in which the sides all have different lengths. She said, "I'll name them after myself." She defined an *alana-gon* to be a quadrilateral in which the four sides all have different lengths. **(1)** Draw an example of an alana-gon. **(2)** True or false: **(a)** All squares are alana-gons. **(b)** No trapezoids are alana-gons.

In later grades, task **3:8 Shape Attributes and Categories** involves reasoning with shape components, shape properties, and shape categories. Spatial structuring is involved in tasks **3:2 Hidden Rug Design**, **3:3 Length and Area Quantities**, and **4:13 Area Units**. Task **4:8 Shapes with Given Positions** involves definitions and properties of geometric figures, and task **5:8 Alana's New Shape Category** involves an invented category defined by attributes.

Additional notes on the design of the task

Correcting an error is one way for primary-grades students to engage in Standard for Mathematical Practice MP.3, "Construct viable arguments and critique the reasoning of others." (CCSS MP.3)

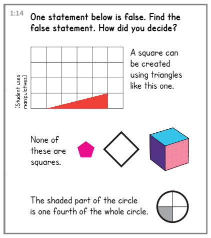
Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 2:14? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 2:14? In what specific ways do they differ from 2:14?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 2:14 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

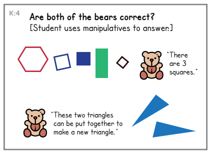
† Common Core Standards Writing Team. (2013, September 19). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona. Page numbers in these Teacher Notes refer to this *Progression*.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

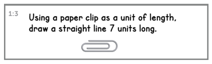
1:14



K:4



1:3



In earlier grades, task **1:14 Shape True/False** involves the same three themes of elementary-grades geometry as task 2:14. In earlier grades, task **K:4 Bears Talk About Shapes** involves shape attributes and composing shapes. Spatial structuring and composing/decomposing are involved in length measurement (iterating length units), as in task **1:3 Paper Clip Length Units**.

2:14 Correcting a Shape Answer
Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?