3:1 Volleyball Players

Teacher Notes

Math Milestones™

Central math concepts

Based on interviews with middle-grades students,† education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract...; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "...if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

Between grades 3 and 5, students' understanding of multiplication evolves from an initial concept of equal groups to a sophisticated concept of scaling. In grade 3, the equal-groups concept of multiplication is that $A \times B$ is the number of things in A groups of B things each. This concept, about what a product expression *means*, must be understood separately from whatever techniques are involved in calculating the *value* of such expressions (what the expression equals; see the Teacher Notes for task **3:2 Hidden Rug Design**). This equal-groups concept also applies to objects arrayed in rows and columns, including rows and columns of unit squares that are arrayed to tile a rectangle (CCSS 3.MD.C.7).

3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?

Equation model: ____________

Answer: ____________

Answer

Equation model: $5 \times 6 = \square$, or an equivalent equation with a symbol for the unknown number, or an equivalent solved equation, such as $5 \times 6 = 30$.
Answer: There are 30 players in the volleyball league.

Click here for a student-facing version of the task.

Refer to the Standards

3.OA.A.3, 3.OA.A; MP.1, MP.2, MP.4.
Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

In indoor volleyball, the teams typically have six players on the court. In outdoor volleyball, teams may have as few as two players on the court. In an informal league, teams may or may not have reserve players.

In grade 4, the idea of equal groups advances to comparative ideas of times-as-many/times-as-much. Simultaneously, fractions enter into the products students calculate; this begins to enlarge the idea of multiplication beyond equal groups of discrete objects. The multiplication problems students solve refer not only to discrete objects like bowling balls or boats, but also substances that can be measured out and repeatedly subdivided, like a quantity of fluid or an interval of time. Often, the measures of these substances are fractions.

In grade 5, ideas of multiplication advance to scaling, including cases in which the product of two factors is smaller than either factor. An essential stage in this learning is understanding the product $\frac{1}{a} \times \frac{1}{b}$ as 1 part of a partition of $\frac{1}{b}$ into a equal parts.

Even in its initial whole-number setting, multiplication is a powerful concept with wide applications. The elementary multiplication and division situations include:‡

- **Equal Groups:** Product Unknown, Group Size Unknown, and Number of Groups Unknown
- **Arrays:** Product Unknown, Number of Rows Unknown, and Number of Columns Unknown
- **Compare:** Size of Larger Quantity Unknown, Size of Smaller Quantity Unknown, and Multiplier Unknown

In particular, the situation type in task 3:1 is called "Equal Groups with Product Unknown." It is an Equal Groups situation because each team has the same number of players. And the situation is "Equal Groups with Product Unknown" because the initially unknown quantity is the total number of players.

Students might determine the value of the unknown product 5 × 6 by remembering it, or they might calculate it by various methods. In the *Progression* document (pp. 25–27), three levels of representation and solution are articulated for multiplying and dividing within the times tables:§

- **Level 1** is making and counting all of the quantities involved in a multiplication or division. Representing the quantities with a diagram affords reflection and sharing when it is drawn on the board and explained by a student.
- **Level 2** is repeated counting on by a given number, such as for 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. The count-bys give the running total. The number of 3s said is tracked with fingers or a visual or physical (e.g., head bobs) pattern. For 8 × 3, you know the number of 3s and count by 3 until you reach 8 of them. For 24 ÷ 3, you count by 3 until you hear 24, then look at your tracking method to see how many 3s you have. Because listening for 24 is easier than monitoring the tracking method for 8 3s to stop at 8, dividing can be easier than multiplying.
- **Level 3** is using the commutative, associative, and/or distributive properties to replace the given problem with an easier problem or subproblems. For example,
  - 4 × 6 can be replaced with 8 × 3:

    $4 \times 6 = 4 \times (2 \times 3) = (4 \times 2) \times 3 = 8 \times 3.$

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:1? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:1? In what specific ways do they differ from 3:1?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:1 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 3, p. 23 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

§ These descriptions of the levels quote from the *Progression* document, with some edits; follow the link for the full *Progression* text.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

- Students may know a product 1 or 2 ahead of or behind a given product and say:

  I know 6 × 5 is 30, so 7 × 5 is 30 + 5 more, which is 35.

  This implicitly uses the distributive property:

  $7 \times 5 = (6 + 1) \times 5 = 6 \times 5 + 1 \times 5 = 30 + 5 = 35.$

- Students may implicitly use the distributive property to decompose a product that they do not know in terms of two products they know (see the Teacher Notes for task **3:10 Alice's Multiplication Fact**). Students may not use the properties explicitly, but classroom discussion can identify and record properties in student reasoning. An area diagram can support such reasoning.

As applied to the product 5 × 6 in task 3:1, a Level 3 (property-based) approach might be that since 6 is 2 3s, then 5 × 6 is 10 3s:

$$\begin{aligned} 5 \times 6 &= 5 \times (2 \times 3) \\ &= (5 \times 2) \times 3 \\ &= 10 \times 3 \\ &= 30. \end{aligned}$$

Task 3:1 asks not only for the final answer but also for an equation model. An equation model is requested because compared to the answer alone ("30 players"), an equation model is better evidence that students have comprehended the situation and its quantitative relationships. The equation model records the situation's mathematical structure so that students can discuss and reflect on it. Not all students may write the same equation. That is an opportunity to discuss correspondences between equations and relationships to the situation described in the problem. The *Progression* document (p. 13) stresses that "what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context."

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: Level 2 and 3 strategies for calculating products; writing equations to describe quantitative relationships; and relating representations such as equations and diagrams to a situation.

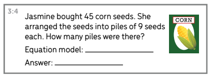
Extending the task

How might students drive the conversation further?

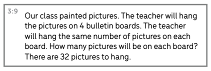
- For each number in an equation model for the situation, such as 5 × 6 = ☐, students could identify the quantity in the situation to which the number refers. For example, 6 refers to "the number of players on each team." Note that naming a quantity is different from naming the numerical value of the quantity.
- Students could determine the total number of players in the league if there are 10 teams. How does the answer relate to the answer to task 3:1?

Related Math Milestones tasks

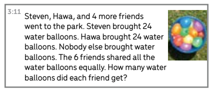
3:4


3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?
Equation model: ___________
Answer: ___________

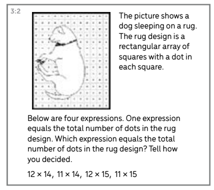
3:9


3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

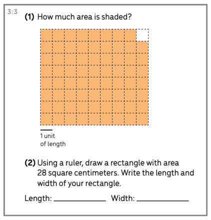
3:11


3:11 Steven, Hawa, and 4 more friends went to the park. Steven brought 24 water balloons. Hawa brought 24 water balloons. Nobody else brought water balloons. The 6 friends shared all the water balloons equally. How many water balloons did each friend get?

3:2

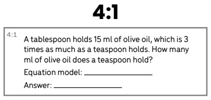

3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.
Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.
12 × 14, 11 × 14, 12 × 15, 11 × 15

3:3

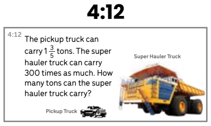

3:3 (1) How much area is shaded?
1 unit of length
(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.
Length: ___________ Width: ___________

Task **3:4 Corn Seeds** is a word problem of type *Equal Groups with Number of Groups Unknown*; task **3:9 Bulletin Board Pictures** is a word problem of type *Equal Groups with Group Size Unknown*; and task **3:11 Water Balloons** is a multi-step word problem involving two situation types, one of which is *Equal Groups with Group Size Unknown*. Task **3:2 Hidden Rug Design** involves interpreting a product expression in an array context. Multiplication is useful in task **3:3 Length and Area Quantities**.

4:1


4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?
Equation model: ___________
Answer: ___________

4:12


4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

In later grades, task **4:1 A Tablespoon of Oil** is a word problem of type *Compare with Size of Smaller Quantity Unknown*, and task **4:12 Super Hauler Truck** is a word problem of type *Compare with Size of Larger Quantity Unknown*.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones].

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:2 Hidden Rug Design

Teacher Notes

Math Milestones™

Central math concepts

Calculating the value of a product of two two-digit numbers isn't part of the expected learning in grade 3. However, even if one doesn't know the value of 12 × 15, one can still represent the total number of dots in the rug design by the expression 12 × 15. That's possible thanks to the fundamental idea behind early multiplication work, which is that for two whole numbers *A* and *B*, the expression *A* × *B* means the total number of objects in *A* groups of *B* objects each. Applying this idea to the rug design, if we think of each row as a group, then there are 15 groups with 12 objects in each group; equivalently, if we think of each column as a group, then there are 12 groups with 15 objects in each group. Taking the second point of view, the number of dots must be equal to 12 x 15—even if we don't know what number that is!

There's an almost magical power in being able to write an expression for the total number of dots without being able to see all of the dots. Wielding that power requires some willingness on the part of the student to view an expression like 12 × 15 not just as a calculation problem ('What is the value of 12 × 15?') but also as a representation problem that involves recognizing the uses and meanings of multiplication. Each factor in the expression 12 × 15 can be connected to particular features of the context at hand, and the expression as whole can also be connected to the context. This is one important way in which elementary-grades students "look for and make use of structure" (CCSS MP.7).

Relevant prior knowledge

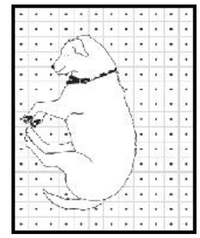
The following mathematics knowledge may be activated, extended, and deepened while students work on the task: spatial structuring and partitioning; rectangular array structures; and fundamental concepts of multiplication with whole numbers.

Extending the task

How might students drive the conversation further?

- Students could choose to satisfy their curiosity by determining the numerical value of 12 × 15 using skip-counting methods or the distributive property. A direct distributive property approach might begin with reasoning that 12 × 15 is ten 15s plus two more 15's, then proceed from there. An implicit distributive property approach might be to break 12 × 15 into a sum, for example 4 × 15 + 4 × 15 + 4 × 15, then work to determine the value of 4 × 15.
- Students could consider a more intricate rug design in which there are 3 dots in each square. Given this design, what is a multiplication expression for the total number of dots?

3:2



The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

12 × 14, 11 × 14, 12 × 15, 11 × 15

Answer

12 × 15. Explanations may vary but should involve the idea that 12 × 15 means the total number of objects in 12 groups of 15 objects each or, equivalently, the total number of objects in 15 groups of 12 objects each. (The number of groups and the number of objects in each group depend on whether we view the array as a collection of columns or a collection of rows.)

Click here for a student-facing version of the task.

Refer to the Standards

3.OA.A.1; MP.2, MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Related Math Milestones tasks

3:1

3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?

Equation model: ________

Answer: ________

3:4

3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?

Equation model: ________

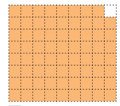
Answer: ________

3:9

3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

3:3

3:3 (1) How much area is shaded?



1 unit of length

(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ________ Width: ________

Tasks **3:1 Volleyball Players**, **3:4 Corn Seeds**, and **3:9 Bulletin Board Pictures** form a kind of survey of the essential early meanings of multiplication and division; the requested equation models can support learning about the relationship between the operations. Multiplication is useful in task **3:3 Length and Area Quantities**.

4:1

4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ________

Answer: ________

4:12

4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

4:10

4:11 A cook in the school kitchen uses 6 oz of cheese to make a pizza. The kitchen has 45 lb of cheese. How many pizzas will that make?

In later grades, tasks **4:1 A Tablespoon of Oil** and **4:12 Super Hauler Truck** involve an important extension of the multiplication concept into times-as-much thinking and multiplicative comparisons. Task **4:10 Calculating Products and Quotients** involves calculating products of two two-digit numbers.

2:2

2:2 (1) True or false?

(a) 2 hundreds + 3 ones > 5 tens + 9 ones

(b) 9 tens + 2 hundreds + 4 ones < 924

(c) 456 < 5 hundreds

(2) Write the number that makes each statement true.

(a) 7 ones + 5 hundreds = ____

(b) 14 tens = ____

(c) 90 + 300 + 4 = ____

In earlier grades, part (1) of task **2:2 Place Value to Hundreds** could be answered by comparing parts of the expressions on either side of the inequality symbol (instead of calculating the values of those expressions).

Additional notes on the design of the task

- It is not the intent that students be able to calculate or estimate the numerical value of the listed products. It is also not the intent that students try to count the dots that are visible, or that students estimate how many dots are hidden by the dog. If students carry out these operations, the results can be gathered as partial knowledge about the situation.
- The importance of the language “rectangular array” in the task is that it guarantees that there is a regular pattern to the dots and their placement, so that students can draw a valid inference about the total number of dots even though some of the dots are hidden by the dog.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:2? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:2? In what specific ways do they differ from 3:2?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:2 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

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Anticipating and responding to student thinking about the task

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Language

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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:3 Length and Area Quantities

Teacher Notes

Central math concepts

In grades K–2, students work with length units and concepts of length measurement (CCSS 1.MD.A.2). Initially in kindergarten, students make non-numerical comparisons of length and other measurable quantities (CCSS K.MD.A). Then in grade 1, using objects as length units, students learn that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. In grade 2, students extend these ideas to abstract length units and relate length measurement to addition, subtraction, and the number line (CCSS 2.MD.A).

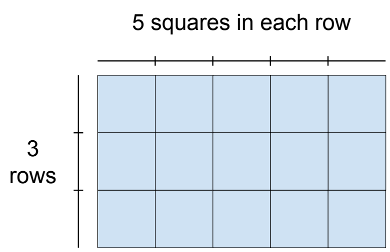
After studying length as a measurable quantity in the primary grades, students learn to recognize area as a measurable attribute of plane figures. Area is the amount of two-dimensional surface a figure contains.

Consistent with this idea, congruent figures are assumed to enclose equal areas. Students also understand the concepts involved in measuring area (CCSS 3.MD.C.5):

- **A unit of measure for area:** An area unit is built from a chosen length unit. Given a length unit, a square with side length equal to 1 unit, called "a unit square," is said to have "one square unit" of area.
- **Quantifying area:** A plane figure that can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units.

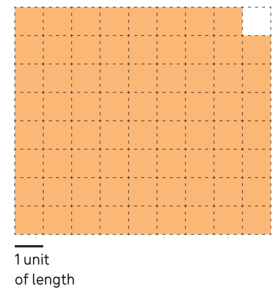
For example, in task 3:3, one perceives that the quantity of shaded area is less, by a single square unit, than it would have been had the entire outer dotted rectangle been shaded.

Area units and length units are closely connected. To see this, imagine tiling a rectangle with unit squares, as shown in the figure. This creates a rectangular array of squares. The number of squares in the array can be found by multiplying the number of squares in each row by the number of rows. An important observation about this figure is that the number of squares in each row of the array equals the number of length units in one of the sides, while the number of rows equals the number of length units in one of the adjacent sides. Thus, the number of squares in the array can be found by multiplying the number of length units in one side by the number of length units in an adjacent side. This conclusion is summarized by the rectangle area formula, $A = L \times W$. In this formula:



- L is the length of one side of the rectangle, measured in length units.
- W is the length of an adjacent side of the rectangle, measured in the same length units that W is measured in.
- A is the number of unit squares needed to tile the rectangle, where a unit square has sides of length equal to 1 length unit (the same length unit that L and W were measured in).

3:3 **(1)** How much area is shaded?

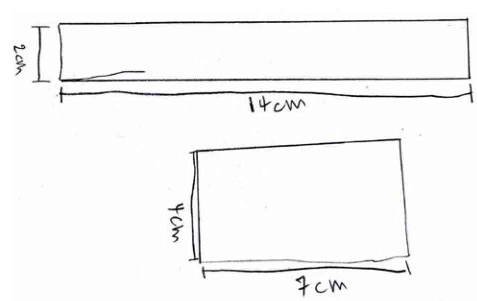


(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ________ Width: ________

Answer

(1) 71 square units. **(2)** See examples. Rectangles that are slightly irregular-looking can still be accurate enough to be considered correct. Rectangles may be rotated at any angle relative to the examples shown. Which side is identified as the length and which side is identified as the width is unimportant, unless these terms have been defined unambiguously in the classroom discourse, in which case the terms should be used as defined. Rectangles such as $3\frac{1}{2}$ cm by 8 cm with one or more fractional side lengths are not expected, but no rectangle is incorrect if its area is 28 square centimeters.



Click here for a student-facing version of the task.

As the above discussion shows, understanding area measurement requires being able to see a blank rectangular region as decomposable into rows and columns of squares. This skill is called spatial structuring (see the Teacher Notes for task **2:14 Correcting a Shape Answer**). In part (1) of task 3:3, the dashed lines provide the spatial structuring.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: spatially structuring a rectangle into an array; decomposing 28 as a product of two factors or as a sum of equal addends; and working with measurement concepts.

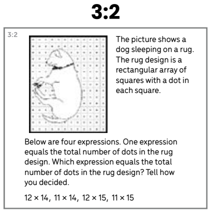
Extending the task

How might students drive the conversation further?

- If some students draw a 4-by-7 rectangle for part (2), while other students draw a 2-by-14 rectangle, students could use grid paper to cut up a 2-by-14 rectangle and show that its parts can exactly cover a 4-by-7 rectangle.
- If students did not show unit squares in their answer for part (2), they could be asked to draw in the unit squares and to verify that there are 28 of them.
- In part (1), if a student found the answer as $9 \times 8 - 1$, then since $9 \times 8 = 8 \times 8 + 8$, the answer could be expressed in a different way as $8 \times 8 + 8 - 1$, which is $8 \times 8 + 7$. Is there a way to connect the expression $8 \times 8 + 7$ to the shape of the shaded region?

Related Math Milestones tasks

3:2

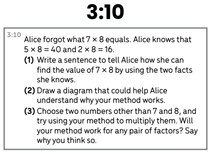

3:2

The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

12×14, 11×14, 12×15, 11×15

3:10


3:10 Alice forgot what 7×8 equals. Alice knows that $5 \times 8 = 40$ and $2 \times 8 = 16$.

(1) Write a sentence to tell Alice how she can find the value of 7×8 by using the two facts she knows.

(2) Draw a diagram that could help Alice understand why your method works.

(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Task **3:2 Hidden Rug Design** relates a product expression to the number of dots in a rectangular array. In part (2) of task **3:10 Alice's Multiplication Fact**, an area diagram could be used to illustrate a distributive property relationship.

Refer to the Standards

3.MD.C; MP.2, MP.5, MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

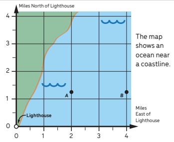
Additional notes on the design of the task

- The statement of part (1) of the task doesn't refer explicitly to a unit of area; this is intended to help reveal the way in which the student talks about or uses the concept of an area unit, when the phrasing of a task doesn't scaffold that use.
- The intent of the relatively large dimensions of the outer dotted rectangle is to establish a situation in which multiplying is more valuable than counting-all.
- Part (2) of the task relates to multiplication in that it involves determining, say, that $28 = 4 \times 7$ or that $28 = 14 + 14 = 2 \times 14$. This element of part (2) is less important, however, than the contemplative act of drawing the rectangle, which creates space for relating the familiar idea of a length unit to the newer idea of an area unit, and which also invites thinking about geometric relationships in a rectangle.
- The drawing in part (2) is preferably made on blank paper, not lined paper or grid paper. This ensures that the lines and grid lines do not clash with the student's plan, and it allows the possibility for the student to do the spatial structuring by drawing lines within the rectangle.

4:13

4:13 **(1)** A red rectangle has length $L = 12$ in and width $W = 6$ in. Use the formula $A = L \times W$ to find the area of the red rectangle.
(2) A blue rectangle has length 1 ft and width $\frac{1}{2}$ ft. Draw a picture to show that two copies of the blue rectangle make one square foot. Based on your picture, what is the area of the blue rectangle?
(3) Do the red rectangle and the blue rectangle have equal areas? Tell how you decided.

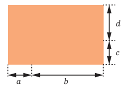
5:7

5:7


The map shows an ocean near a coastline.

Shipwrecks are at locations $A\ (2, 1\frac{1}{4})$ and $B\ (4, 1\frac{1}{4})$. Shipwrecks are also at locations $C\ (4, 3\frac{1}{2})$ and $D\ (2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

6:11

6:11 The diagram shows a rectangle. The variables a, b, c, and d are lengths in meters.


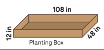
(1) Using the variables, write three different expressions for the area of the rectangle. **(2)** Choose two of your expressions and show that they are equivalent by applying properties of operations. **(3)** State the property or properties you used.

6:12

6:12 **(1)** What is the area of the triangle in the coordinate plane with vertices (1, 2), (−5, 2), and (−8, 9)? **(2)** How does the area change if we change the third vertex to (−3, 9)?

5:3

5:3 A neighborhood garden will have 6 wooden planting boxes. Every box will have the same shape (see diagram). Soil can be bought by the truckload; a truckload is 54 ft^3 of soil. How many truckloads of soil will fill all of the boxes?



8:12

8:12 Design a fish tank that fits into the corner of a room. Use a quarter of a cylinder as a model for the tank. To share your design, make a diagram showing the tank measurements. Also, calculate the weight of the water when your tank is filled (1 m^3 of water weighs about 1,000 kg). Write your calculation steps so that a classmate could understand how you did it.


In later grades, task **4:13 Area Units** involves a quantity of area measured using two different area units. Task **5:7 Shipwrecks** involves rectangle area in context for a rectangle with fractional dimensions, and task **6:11 Area Expressions** involves rectangle area in a case where the lengths are variables rather than numbers. Task **6:12 Coordinate Triangle** involves area measure for a triangle. Tasks **5:3 Neighborhood Garden** and **8:12 Fish Tank Design** are applications involving volume measurement.

1:3

1:3 Using a paper clip as a unit of length, draw a straight line 7 units long.


2:14

2:14 Zariah got one answer wrong.
(1) Which answer did Zariah get wrong?
(2) Correct Zariah's wrong answer.

(a) Show how the rectangle can be divided into 15 squares.


(b) 2 halves make one whole.

(c) Draw a triangle. All three sides of your triangle must have different lengths.


In earlier grades, task **1:3 Paper Clip Length Units** involves creating a length from units. Task **2:14 Correcting a Shape Answer** (part (2)(a)) involves spatial structuring.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:3? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:3? In what specific ways do they differ from 3:3?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:3 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:4 Corn Seeds

Teacher Notes

Central math concepts

Based on interviews with middle-grades students,[†] education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

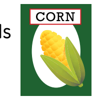
The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations, so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

Problems involving division also implicitly involve multiplication, because division finds an unknown factor. That is, $C \div A$ is the unknown factor in $A \times \square = C$. This is why a division calculation is checked by multiplying. From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between multiplication and division can play out in situations in conceptually distinct ways, including:[‡]

- **Equal Groups:** Product Unknown, Group Size Unknown, and Number of Groups Unknown

3:4

Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?



Equation model: ______________

Answer: ______________

Answer

Equation model: $\square \times 9 = 45$, $45 \div 9 = \square$, or an equivalent equation with a symbol for the unknown number, or an equivalent solved equation, such as $5 \times 9 = 45$. Answer: There were 5 piles of seeds.

Click here for a student-facing version of the task.

Refer to the Standards

3.OA.A.3, 3.OA.A; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

The reason the seeds in the task are corn seeds is that many other kinds of garden seeds (such as lettuce seeds) are very small, and those kinds of seeds would be difficult to handle one by one.

- **Arrays:** Product Unknown, Number of Rows Unknown, and Number of Columns Unknown
- **Compare:** Size of Larger Quantity Unknown, Size of Smaller Quantity Unknown, and Multiplier Unknown

In particular, the situation type in task 3:4 is called "Equal Groups with Number of Groups Unknown." It is an Equal Groups situation because each pile contains the same number of seeds. And the situation is "Equal Groups with Number of Groups Unknown" because the initially unknown quantity is the number of piles of seeds.

Students might determine the value of the unknown factor in $\square \times 9 = 45$ by remembering that $5 \times 9 = 45$, or they might calculate the quotient $45 \div 9 = 5$ by various methods. In the *Progression* document (pp. 25–27), three levels of representation and solution are articulated for multiplying and dividing within the times tables:§

- **Level 1** is making and counting all of the quantities involved in a multiplication or division. Representing the quantities with a diagram affords reflection and sharing when it is drawn on the board and explained by a student.
- **Level 2** is repeated counting on by a given number, such as for 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. The count-bys give the running total. The number of 3s said is tracked with fingers or a visual or physical (e.g., head bobs) pattern. For 8×3, you know the number of 3s and count by 3 until you reach 8 of them. For $24 \div 3$, you count by 3 until you hear 24, then look at your tracking method to see how many 3s you have. Because listening for 24 is easier than monitoring the tracking method for 8 3s to stop at 8, dividing can be easier than multiplying.
- **Level 3** is using the commutative, associative, and/or distributive properties to replace the given problem with an easier problem or subproblems. For example,
  - 4×6 can be replaced with 8×3:

    $4 \times 6 = 4 \times (2 \times 3) = (4 \times 2) \times 3 = 8 \times 3.$
  - Students may know a product 1 or 2 ahead of or behind a given product and say:

    I know 6 × 5 is 30, so 7 × 5 is 30 + 5 more, which is 35.

    This implicitly uses the distributive property:

    $7 \times 5 = (6 + 1) \times 5 = 6 \times 5 + 1 \times 5 = 30 + 5 = 35.$
  - Students may implicitly use the distributive property to decompose a product that they do not know in terms of two products they know (see the Teacher Notes for task **3:10 Alice's Multiplication Fact**). Students may not use the properties explicitly, but classroom discussion can identify and record properties in student reasoning. An area diagram can support such reasoning.

As applied to the quotient $45 \div 9$ in task 3:4, a Level 3 (property-based) approach might be that if we know 3 9s are 27, then 27 is less than 45 by an amount $3 + 10 + 5 = 18$, and that's 2 9s, so 45 is 5 9s. This approach implicitly uses the distributive property as follows:

$27 + 18 = 45$

$3 \times 9 + 2 \times 9 = 45$

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:4? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:4? In what specific ways do they differ from 3:4?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:4 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report.* San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 3, p. 23 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

§ These descriptions of the levels quote from the *Progression* document, with some edits; follow the link for the full *Progression* text.

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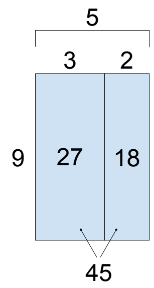
$(3 + 2) \times 9 = 45$

$5 \times 9 = 45.$

The area diagram illustrates aspects of this calculation.



Another approach might be to think that since $90 \div 9 = 10$, then $45 \div 9 = 10 \div 2$:

$$\begin{aligned} 45 \div 9 &= 2 \times (45 \div 9) \div 2 \\ &= (90 \div 9) \div 2 \\ &= 10 \div 2 \\ &= 5. \end{aligned}$$

Task 3:4 asks not only for the final answer but also for an equation model. An equation model is requested because compared to the answer alone ("5 piles of seeds"), an equation model is better evidence that students have comprehended the situation and its quantitative relationships. The equation model records the situation's mathematical structure so that students can discuss and reflect on it. Equivalent forms of the equation can also show the relationship between multiplication and division.

It may happen that not all students write the same equation. That is an opportunity to discuss correspondences between equations and relationships to the situation described in the problem. The *Progression* document stresses that "what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context" (p. 13).

Some equation models describe a situation in an algebraic way, such as $\square \times 9 = 45$, (these are called *situation equations*); students may then rewrite a situation equation in a form more conducive to determining the unknown number, such as $45 \div 9 = \square$ (this is called a *solution equation*). As observed in the *Progression* document,

> Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. (p. 13)

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: Level 2 and 3 strategies for calculating products and quotients; finding quotients by remembering products; writing equations to describe quantitative relationships; and relating representations such as equations and diagrams to a situation.

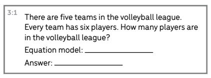
Extending the task

How might students drive the conversation further?

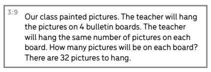
- For each number in an equation model for the situation, such as □ × 9 = 45, students could identify the quantity in the situation to which the number refers. For example, 45 refers to “the total number of seeds.” Note that naming a quantity is different from naming the numerical value of the quantity.
- Students could check their quotient by multiplying.

Related Math Milestones tasks

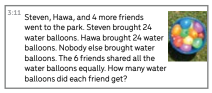
3:1


3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?
Equation model: ____________
Answer: ____________

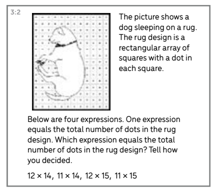
3:9


3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

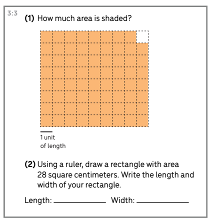
3:11


3:11 Steven, Hawa, and 4 more friends went to the park. Steven brought 24 water balloons. Hawa brought 24 water balloons. Nobody else brought water balloons. The 6 friends shared all the water balloons equally. How many water balloons did each friend get?

3:2


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.
Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.
12 × 14, 11 × 14, 12 × 15, 11 × 15

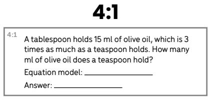
3:3


3:3 (1) How much area is shaded?

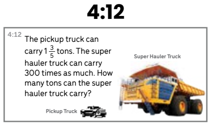
(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.
Length: ____________ Width: ____________

Task **3:1 Volleyball Players** is a word problem of type *Equal Groups with Product Unknown*; task **3:9 Bulletin Board Pictures** is a word problem of type *Equal Groups with Group Size Unknown*; and task **3:11 Water Balloons** is a multi-step word problem involving two situation types, one of which is *Equal Groups with Group Size Unknown*. Task **3:2 Hidden Rug Design** involves interpreting a product expression in an array context. Multiplication is useful in task **3:3 Length and Area Quantities**.

4:1


4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?
Equation model: ____________
Answer: ____________

4:12


4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

In later grades, task **4:1 A Tablespoon of Oil** is a word problem of type *Compare with Size of Smaller Quantity Unknown*, and task **4:12 Super Hauler Truck** is a word problem of type *Compare with Size of Larger Quantity Unknown*.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones].

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:5 Playground Cleanup

Teacher Notes

Central math concepts

In grade 2, students drew picture graphs in which each picture represents one object, and they drew bar graphs with a single-unit scale. In grade 3, as explained in the relevant *Progression* document,[†]

> the most important development in data representation for categorical data is that students now draw picture graphs in which each picture represents more than one object, and they draw bar graphs in which the height of a given bar in tick marks must be multiplied by the scale factor in order to yield the number of objects in the given category. (p. 7)

The *Progression* document also notes that "[t]hese developments connect with the emphasis on multiplication in this grade" (p. 7). More generally, there are close connections in every elementary grade between students' data work and their expanding use of numbers and operations in context; see Table 1, p. 4 for a list of these connections in grades K–5.

Constructing a picture graph or a bar graph involves identifying quantities in the situation, specifying units of measure, and attending to precision. To support students in creating picture graphs and bar graphs, grid paper is useful. As noted in the *Progression* document (p. 7), "When drawing picture graphs on grid paper, the pictures representing the objects should be drawn in the squares of the grid paper." For bar graphs, "the tick marks on the count scale should be drawn at intersections of the gridlines. The tops of the bars should reach the respective gridlines of the appropriate tick marks." For both kinds of graphs, as suggested in the *Progression* document (p. 3), "a template can be superimposed on the grid paper, with blanks provided for the student to write in the graph title, scale labels, category labels, legend, and so on."

As the *Guidelines for Assessment and Instruction in Statistics Education Report* notes, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning."[‡] Thus as the *Progression* document notes (p. 3), "students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the context they represent." That said, "students do not have to generate the data every time they work on making bar graphs and picture graphs. That would be too time-consuming. After some experiences in generating the data, most work in producing bar graphs and picture graphs can be done by providing students with data sets" (p. 7).

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: finding a total number for groups of 5 or 10 using Level 2 count-by strategy; and using known single-digit products involving factors of 5.

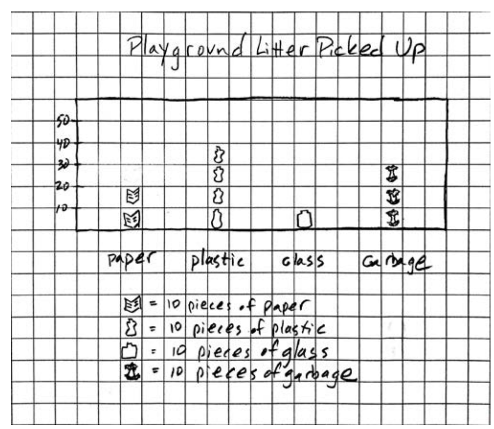
3:5

Our class picked up litter on the playground. One student wrote tally marks to record the things we picked up.

Show the data another way by drawing a scaled picture graph in which 1 picture stands for 10 things picked up.

Answer

See example. The graph should include a title, category labels, and a legend. Ticks on the vertical count scale are optional. The order of the categories doesn't matter. The spacing between the category bars doesn't matter. The graph may be oriented horizontally or vertically. Pictures representing the categories need not be representational (for example, they could be stars, diamonds, circles, and X's). Wording may vary for the title and the legend, but precision is desirable in naming quantities in the legend.



Click here for a student-facing version of the task.

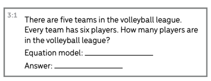
Extending the task

How might students drive the conversation further?

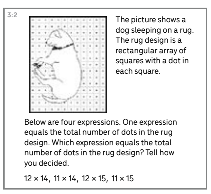
- Students could discuss how the scale for the picture graph communicates what happened during the playground cleanup. Would the graph better communicate the volume of effort if 1 picture stood for 5 things picked up? What would be the advantages and disadvantages of a scale in which 1 picture stands for 1 thing picked up?
- Students could create a bar graph of the same data and compare the way it represents the data to the way the picture graph represents the data. Are there advantages to each?
- Students could recategorize the data in two categories, Recyclable items and Garbage items, treating Paper, Plastic, and Glass as recyclable. How many items are in the Recyclable category? What would the bar graph look like for the categories of Recyclable and Garbage, compared to the one with categories of Paper, Plastic, Glass, and Garbage?

Related Math Milestones tasks

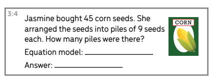
3:1


3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?
Equation model: ______
Answer: ______

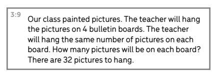
3:2


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.
Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.
12 × 14, 11 × 14, 12 × 15, 11 × 15

3:4

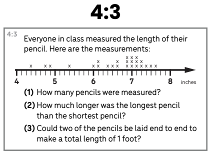

3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?
Equation model: ______
Answer: ______

3:9


3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

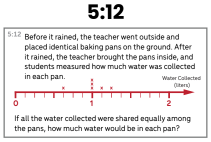
Tasks **3:1 Volleyball Players**, **3:2 Hidden Rug Design**, **3:4 Corn Seeds**, and **3:9 Bulletin Board Pictures** involve multiplication and division contexts.

4:3


4:3 Everyone in class measured the length of their pencil. Here are the measurements:

(1) How many pencils were measured?
(2) How much longer was the longest pencil than the shortest pencil?
(3) Could two of the pencils be laid end to end to make a total length of 1 foot?

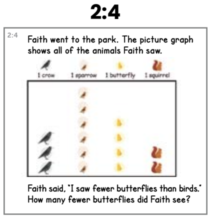
5:12


5:12 Before it rained, the teacher went outside and placed identical baking pans on the ground. After it rained, the teacher brought the pans inside, and students measured how much water was collected in each pan.

If all the water collected were shared equally among the pans, how much water would be in each pan?

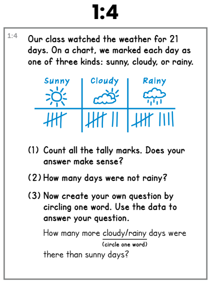
In later grades, tasks **4:3 Pencil Data** and **5:12 Rain Measurements** involve problem solving based on measurement data displayed in line plots.

2:4


2:4 Faith went to the park. The picture graph shows all of the animals Faith saw.

Faith said, "I saw fewer butterflies than birds." How many fewer butterflies did Faith see?

1:4


1:4 Our class watched the weather for 21 days. On a chart, we marked each day as one of three kinds: sunny, cloudy, or rainy.

(1) Count all the tally marks. Does your answer make sense?
(2) How many days were not rainy?
(3) Now create your own question by circling one word. Use the data to answer your question.
How many more cloudy/rainy (circle one word) days were there than sunny days?

K:14


K:14 Are there more land animals or more sea animals?

In earlier grades, tasks **2:4 Animals in the Park**, **1:4 Analyzing Weather Data**, and **K:14 Animals from Land and Sea** involve problem solving based on categorical data.

Refer to the Standards

3.MD.B.3; MP.2, MP.4, MP.6. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

- The tally marks in task 3:5 are shown in groups of 5, and the bar graph is requested in units of 10, so as to provide connections with multiplication and division work in this grade.
- Three of the four categories refer to materials that are often recyclable (paper, plastic, and glass).

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:5? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:5? In what specific ways do they differ from 3:5?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:5 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2011, June 20). *Progressions for the Common Core State Standards in Mathematics (draft): K–3, Categorical Data; Grades 2–5, Measurement Data* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ The Guidelines for Assessment and Instruction in Statistics Education Report was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

3:5 Playground Cleanup

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

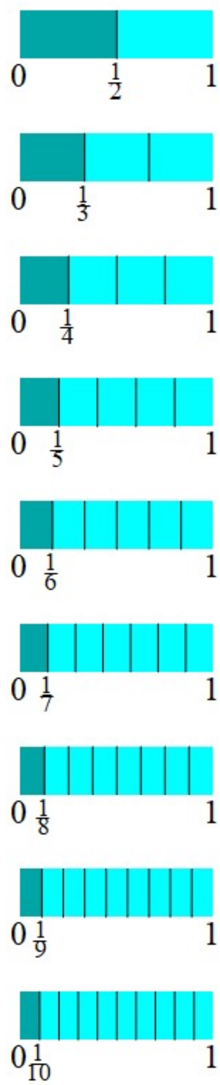
3:6 Unit Fraction Ideas

Teacher Notes

Math Milestones™

Central math concepts

Possibly the two most important foundations for fraction arithmetic are the idea of the unit fraction and the principle of fraction equivalence. Students will frequently employ unit fraction ideas and the principle of fraction equivalence throughout the upper-elementary grades as they learn to extend arithmetic from whole numbers to fractions. Task 3:6 invests in the first of these essential concepts, the unit fraction.



Unit fractions play an important role in school mathematics for at least three reasons. One is that everyday life commonly presents us with situations in which a measurement unit is too large to serve our purposes. For example, when hanging pictures on a wall, measuring to the nearest whole inch might not be precise enough, so we may choose to work with smaller units such as fourths or eighths. A second reason for the importance of unit fractions is that unit fractions allow students to apply unit thinking to fraction problems. As a grade 6 example, the quotient $\frac{6}{7} \div \frac{2}{7}$ can be seen to equal 3 for the same reason that 6 of any unit divided by 2 of that same unit equals 3. In this case, the units are *sevenths*.

A third reason why unit fractions are so important in school mathematics is the way they unify multiplication and division. Think for example about the idea that $\frac{4}{5}$ is the quantity formed by 4 parts when a whole is partitioned into 5 equal parts. That sounds like multiplication (because there are 4 of something), but it also sounds like division (because we are partitioning). By the upper elementary grades, fractions will be breaking down conceptual walls between multiplication and division. For example, dividing by B can be accomplished just as well by multiplying by $\frac{1}{B}$.†

When talking about fractions, it is desirable to talk as much as possible about *fractions*, not about whole numbers. One way to do this is to refer frequently to the units of thirds, fourths, fifths, etc., when possible, instead of referring only to denominators 3, 4, 5, etc. "Thirds are smaller than halves" is an important piece of quantitative thinking. A mnemonic device such as "To compare unit fractions, compare the denominators" contains much less mathematics and could be easy to misapply. Nevertheless, a general understanding of how the size of a unit fraction depends on the denominator is a valuable insight and a kind of early functional thinking.

3:6 Using what you know about fractions, decide which is greater, $\frac{1}{73}$ or $\frac{1}{41}$. Tell how you decided.

Answer

$\frac{1}{41}$ is greater than $\frac{1}{73}$. Explanations may vary. One kind of explanation involves the idea that if you partition a whole into a greater number of equal parts, then the parts have smaller size than when you partition a whole into a lesser number of equal parts. This might be illustrated with contexts, such as saying that one would rather receive $\frac{1}{41}$ of a bar of gold than $\frac{1}{73}$ of the bar of gold, because when you slice a bar of gold into 41 equal parts, those parts are larger (heavier) than when you slice the bar into 73 equal parts.

Click here for a student-facing version of the task.

Refer to the Standards

3.NF.A; MP.1, MP.5, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: spatial structuring and partitioning; unit ideas in measurement; and basing reasoning on math diagrams.

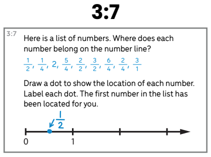
Extending the task

How might students drive the conversation further?

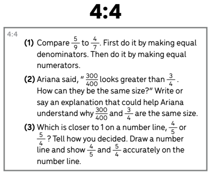
- Students could invent even "sillier" versions of the problem, such as the problem of comparing $\frac{1}{1{,}000{,}000}$ to $\frac{1}{1{,}000{,}001}$.
- Students could propose a mathematical discovery, for example by realizing that there is no smallest number. One justification for this is that a whole can be partitioned into any number of parts no matter how many.

Related Math Milestones tasks

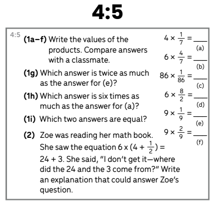
3:7



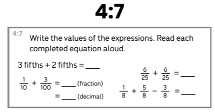
4:4



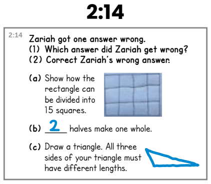
4:5



4:7



2:14



Task **3:7 Locating Numbers on a Number Line** integrates concepts of unit fractions, whole numbers, and simple cases of fraction equivalence.

In later grades, task **4:4 Comparing Fractions with Equivalence** takes up the second fundamental fraction concept mentioned in "Central math concepts." Task **4:5 Fraction Products and Properties** extends third-grade quantitative thinking about fractions into multiplication, while task **4:7 Fraction Sums and Differences** does something similar for addition, emphasizing the role of unit fractions.

In earlier grades, task **2:14 Correcting a Shape Answer** involves structuring space by partitioning a rectangle and relating partitions to fractional parts (halves).

Additional notes on the design of the task

- Task 3:6 isn't routine; the denominators in the fractions are much greater than those which students commonly study. Drawing fraction models for simpler cases can be useful while investigating the problem, although by design it is unlikely that a single drawing in itself could convincingly settle the question being asked.
- The intent of the task is for students to expand and deepen their understanding through investigation and contemplation. The task is not designed as an opportunity to simply apply a remembered mnemonic or procedure. A general realization about how the size of a unit fraction depends on the size of the numerator can emerge from the investigation and be celebrated as a discovery.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:6? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:6? In what specific ways do they differ from 3:6?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:6 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Fractions make division superfluous from an axiomatic point of view; one may notice that in the properties of operations that govern algebra, there are no properties that refer to division. Division thinking remains important for modeling and problem-solving, even for algebraic problems; the point of principle is that mathematically speaking, any quotient could be rewritten as a product, because of the existence of multiplicative inverses. (The unit fractions in particular are the multiplicative inverses of the nonzero whole numbers.)

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:7 Locating Numbers on a Number Line

Teacher Notes

Central math concepts

As noted in the CCSS Standards, a student's conception of number evolves tremendously during the K–8 years:

> At first, "number" means "counting number": 1, 2, 3, ... Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.
>
> This ascent through number systems makes it fair to ask: what does the word number mean that it can mean all of these things? One possible answer is that a number is something that can be used to do mathematics: calculate, solve equations, or represent measurements. (p. 58)

The work students do in grade 3 to understand fractions as numbers represents a major development in their mathematical progress. Because fractions have a different format from whole numbers, it may not be apparent that whole numbers and fractions belong to a single family of numbers. But the whole numbers are embedded in the fractions, because $0 = \frac{0}{1}$, $1 = \frac{1}{1}$, $2 = \frac{2}{1}$, $3 = \frac{3}{1}$, and so on. The number line is a useful diagram for representing fractions and whole numbers together as a single family of numbers.

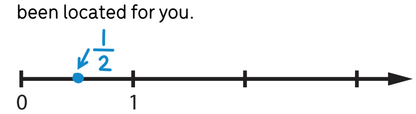
Everyday language works against mathematical language in this area, because in everyday language, "fraction" often carries a connotation of "less than one." (When we say that one purchasing option is "a fraction of the cost" of another option, we don't mean a fraction greater than one.) The everyday connotation of "fraction" as "less than one" could make it counterintuitive that fractions can be greater than one or equal to whole numbers.

The number line is useful for showing equivalences, such as the equivalences $\frac{2}{4} = \frac{1}{2}$ and $\frac{2}{2} = 1$ that appear in task 3:7. Two fractions are equal if they are the same size, or the same point on a number line. In grade 4, this idea leads to the general equivalence principle $\frac{a}{b} = \frac{n \times a}{n \times b}$, which plays a significant role in applying and extending addition, subtraction, multiplication, and division from whole numbers to fractions.

3:7 Here is a list of numbers. Where does each number belong on the number line?

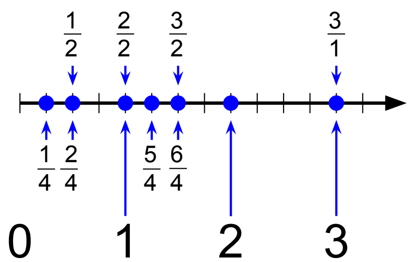
$\frac{1}{2}, \frac{1}{4}, 2, \frac{5}{4}, \frac{2}{2}, \frac{3}{2}, \frac{6}{4}, \frac{2}{4}, \frac{3}{1}$

Draw a dot to show the location of each number. Label each dot. The first number in the list has been located for you.



Answer

See diagram.



Click here for a student-facing version of the task.

Refer to the Standards

3.NF.A; MP.1, MP.2, MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- The task includes two cases in which a fraction equals a whole number: one in the form $\frac{n}{n} = 1$ and another in the form $\frac{n}{1} = n$.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: understanding unit fractions $\frac{1}{2}$ and $\frac{1}{4}$; and using a number line to support and communicate mathematical reasoning.

Extending the task

How might students drive the conversation further?

- Students could discuss and compare the different ways they located each point.
- Students could express 2 as a fraction with a denominator of 2 by continuing the count along the top of the diagram ($\frac{1}{2}, \frac{2}{2}, \frac{3}{2}$, ...), or express 2 as a fraction with a denominator of 4 by continuing the count below the number line ($\frac{5}{4}, \frac{6}{4}$, ...).
- Students could use unit fraction thinking or a story context to make sense of the fact that $\frac{1}{4}$ and $\frac{2}{4}$ are closer together on the number line than $\frac{1}{2}$ and $\frac{2}{2}$.

Related Math Milestones tasks

3:6

3:6 Using what you know about fractions, decide which is greater, $\frac{1}{73}$ or $\frac{1}{41}$. Tell how you decided.

4:4

4:4
(1) Compare $\frac{5}{9}$ to $\frac{4}{7}$. First do it by making equal denominators. Then do it by making equal numerators.
(2) Ariana said, "$\frac{300}{400}$ looks greater than $\frac{3}{4}$. How can they be the same size?" Write or say an explanation that could help Ariana understand why $\frac{300}{400}$ and $\frac{3}{4}$ are the same size.
(3) Which is closer to 1 on a number line, $\frac{4}{5}$ or $\frac{5}{4}$? Tell how you decided. Draw a number line and show $\frac{4}{5}$ and $\frac{5}{4}$ accurately on the number line.

4:5

4:5
(1a–f) Write the values of the products. Compare answers with a classmate.
(1g) Which answer is twice as much as the answer for (e)?
(1h) Which answer is six times as much as the answer for (a)?
(1i) Which two answers are equal?
(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

$4 \times \frac{1}{7} =$ ___ (a)
$6 \times \frac{4}{7} =$ ___ (b)
$86 \times \frac{1}{86} =$ ___ (c)
$6 \times \frac{8}{2} =$ ___ (d)
$9 \times \frac{1}{9} =$ ___ (e)
$9 \times \frac{2}{9} =$ ___ (f)

4:7

4:7 Write the values of the expressions. Read each completed equation aloud.

3 fifths + 2 fifths = ___

$\frac{1}{10} + \frac{3}{100} =$ ___ (fraction)
= ___ (decimal)

$\frac{6}{25} + \frac{6}{25} =$ ___

$\frac{1}{8} + \frac{5}{8} - \frac{3}{8} =$ ___

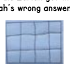
5:10

5:10
(1) Solve: $\frac{1}{3} = 0.1 + ?$
(2) Is there a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$? If you think so, find such a number. If you think there is no such number, explain why.
(3) Show one of the above problems and its solution on a number line.

2:14

2:14 Zariah got one answer wrong.
(1) Which answer did Zariah get wrong?
(2) Correct Zariah's wrong answer.

(a) Show how the rectangle can be divided into 15 squares.



(b) 2 halves make one whole.

(c) Draw a triangle. All three sides of your triangle must have different lengths.

2:11

2:11 A grass snake is 28 inches long. A rat snake is 74 inches long. How much longer is the rat snake?
Draw a diagram to illustrate your solution. Label the diagram with numbers.

Task **3:6 Unit Fraction Ideas** involves the foundational fraction concept, the unit fraction.

In later grades, fraction equivalence and fraction size are the focus in task **4:4 Comparing Fractions with Equivalence**. Operations begin to be extended from whole numbers to fractions in tasks **4:5 Fraction Products and Properties** and **4:7 Fraction Sums and Differences**. Task **5:10 Number System, Number Line** involves fractions and decimals together on the number line.

In earlier grades, task **2:14 Correcting a Shape Answer** involves the concept of one-half as well as spatial structuring that underlies later work with area models. Number lines are useful as diagrams for addition and subtraction problems involving length, for example task **2:11 Grass Snake vs. Rat Snake**.

Additional notes on the design of the task (continued)

- Students don't have to recognize cases of equivalence in advance; the equivalence can emerge from locating a dot where a dot had previously been placed.
- There are many fractions to locate, and this is intended not only to cover several important cases and equivalences but also to create possibilities for using previously located points to inform the choice of where to locate subsequent points. The large number of fractions also begins to suggest the plenitude of numbers themselves (see the Teacher Notes for task **5:10 Number System, Number Line**).

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:7? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:7? In what specific ways do they differ from 3:7?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:7 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† See Zimba (2013), "Units, a Unifying Idea in Measurement, Fractions, and Base Ten."

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
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Language

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- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:8 Shape Attributes and Categories

Teacher Notes

Math Milestones™

Central math concepts

A distinct thread of geometry learning in the elementary grades deals with shape components, shape properties, and shape categories. The *Progression document*† describes three *levels of geometric thinking* that describe increasing sophistication with this learning progression:

- **Visual/Syncretic level**. Students recognize shapes, for example, a rectangle "looks like a door."
- **Descriptive level**. Students perceive properties of shapes, for example, a rectangle has four sides, all its sides are straight, opposite sides have equal length.
- **Analytic level**. Students characterize shapes by their properties, for example, a rectangle has opposite sides of equal length and four right angles.
- **Abstract level**. Students understand, for example, that a rectangle is a parallelogram because it has all the properties of parallelograms.

By grade 3, as noted in the *Progression* document (p. 13), categories of shapes "can be the raw material for thinking about the relationships between classes. For example, students can form larger, superordinate, categories, such as the class of all shapes with four sides, or quadrilaterals, and recognize that it includes other categories, such as squares, rectangles, rhombuses, parallelograms, and trapezoids. They also recognize that there are quadrilaterals that are not in any of those subcategories." Task 3:8 concentrates on these general notions of shape classifications and relationships between categories of shapes.

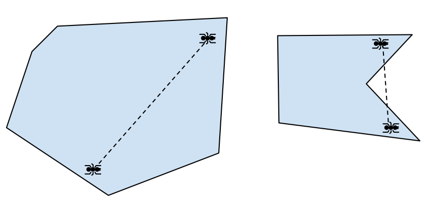
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: remembering definitions of polygons or other two-dimensional shapes; identifying and naming shape attributes; and drawing examples of shapes with defined properties or in defined categories.

Extending the task

How might students drive the conversation further?

- Students could decide, for a given two-dimensional shape, if it is possible for an imaginary tiny ant to travel from any point in the shape to any other point in the shape without ever being outside the shape. In the figure, the shape on the left has this property, but the shape on the right does not.



3:8 **(1)** Name two attributes that are shared by triangles and squares.
(2) Name a category of shapes that includes triangles and squares and also includes other shapes that have both of the attributes you named.

Answer

(1) Answers may vary, but could include such observations as: triangles and squares both have straight sides; triangles and squares are both closed; triangles and squares are both two-dimensional. **(2)** Possible answers include the category of polygons (which includes hexagons, for example); the category of closed two-dimensional shapes (which includes circles, for example); the category of polygons with fewer than 5 sides (which includes kites, for example).

Click here for a student-facing version of the task.

Refer to the Standards

3.G.A.1; MP.1, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

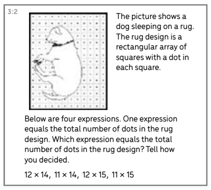
Additional notes on the design of the task

Part (1) of the task directs attention to attributes; part (2) uses the attributes to direct attention to categories.

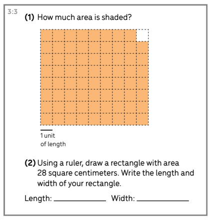
This property defines a category of shapes. What might we call this shape category?‡ Which familiar shapes belong to the category? Could we draw some more shapes that don't belong to the category? Does the category make sense for shapes with curved sides?

Related Math Milestones tasks

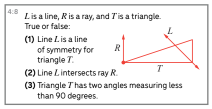
3:2



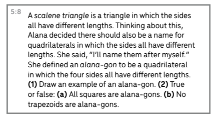
3:3



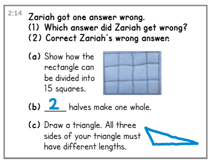
4:8



5:8



2:14



Geometric properties of rectangles are implicit when reasoning about arrays and area; arrays are the context of task **3:2 Hidden Rug Design**, and area is the context of task **3:3 Length and Area Quantities**.

In later grades, task **4:8 Shapes with Given Positions** involves definitions of geometric figures, and task **5:8 Alana's New Shape Category** involves an invented category defined by attributes.

In earlier grades, task **2:14 Correcting a Shape Answer** involves creating a shape with a specified attribute.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:8? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:8? In what specific ways do they differ from 3:8?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:8 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2013, September 19). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona, p. 3.

‡ Mathematicians call this category "convex shapes," but students could invent a name for the category, even a whimsical one, based on their understanding of the property.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:9 Bulletin Board Pictures

Teacher Notes

Central math concepts

Based on interviews with middle-grades students,† education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

Problems involving division also implicitly involve multiplication, because division finds an unknown factor. That is, $C \div A$ is the unknown factor in $A \times ? = C$. This is why a division calculation is checked by multiplying. From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between multiplication and division can play out in situations in conceptually distinct ways, including:

- **Equal Groups:** Product Unknown, Group Size Unknown, and Number of Groups Unknown

3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

Answer

8 pictures will be on each board.

Click here for a student-facing version of the task.

Refer to the Standards

3.OA.A.3, 3.OA.A; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

The text of the problem deviates from the norm in that the value of a quantity in the situation is provided after the question has been posed.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:9? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:9? In what specific ways do they differ from 3:9?

- **Arrays:** Product Unknown, Number of Rows Unknown, and Number of Columns Unknown
- **Compare:** Size of Larger Quantity Unknown, Size of Smaller Quantity Unknown, and Multiplier Unknown

In particular, the situation type in task 3:9 is called "Equal Groups with Group Size Unknown." It is an Equal Groups situation because each bulletin board will hold the same number of pictures. And the situation is "Equal Groups with Group Size Unknown" because the initially unknown quantity is the number of pictures on each bulletin board.

Students might determine the value of the unknown factor in $4 \times \square = 32$ by remembering that $4 \times 8 = 32$, or they might calculate the quotient $32 \div 4 = 8$ by various methods. In the *Progression* document (pp. 25–27), three levels of representation and solution are articulated for multiplying and dividing within the times tables:[§]

- **Level 1** is making and counting all of the quantities involved in a multiplication or division. Representing the quantities with a diagram affords reflection and sharing when it is drawn on the board and explained by a student.
- **Level 2** is repeated counting on by a given number, such as for 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. The count-bys give the running total. The number of 3s said is tracked with fingers or a visual or physical (e.g., head bobs) pattern. For 8×3, you know the number of 3s and count by 3 until you reach 8 of them. For $24 \div 3$, you count by 3 until you hear 24, then look at your tracking method to see how many 3s you have. Because listening for 24 is easier than monitoring the tracking method for 8 3s to stop at 8, dividing can be easier than multiplying.
- **Level 3** is using the commutative, associative, and/or distributive properties to replace the given problem with an easier problem or subproblems. For example,
  - 4×6 can be replaced with 8×3:

    $4 \times 6 = 4 \times (2 \times 3) = (4 \times 2) \times 3 = 8 \times 3.$
  - Students may know a product 1 or 2 ahead of or behind a given product and say:

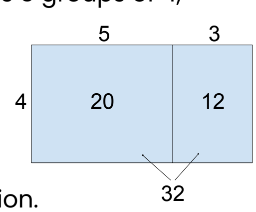
    I know 6×5 is 30, so 7×5 is $30 + 5$ more, which is 35.

    This implicitly uses the distributive property:

    $7 \times 5 = (6 + 1) \times 5 = 6 \times 5 + 1 \times 5 = 30 + 5 = 35.$
  - Students may implicitly use the distributive property to decompose a product that they do not know in terms of two products they know (see the Teacher Notes for task **3:10 Alice's Multiplication Fact**). Students may not use the properties explicitly, but classroom discussion can identify and record properties in student reasoning. An area diagram can support such reasoning.

As applied to the quotient $32 \div 4$ in task 3:9, a Level 3 (property-based) approach might be to think of 32 as 20 + 12: then 20 is 5 groups of 4, and 12 is 3 groups of 4, so 32 is 8 groups of 4:

$$\begin{aligned} 32 \div 4 &= 20 \div 4 + 12 \div 4 \\ &= 5 + 3 \\ &= 8. \end{aligned}$$



The area diagram illustrates aspects of this calculation.

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:9 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report.* San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 3, p. 23 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

§ These descriptions of the levels quote from the *Progression* document, with some edits; follow the link for the full *Progression* text.

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Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: Level 2 and 3 strategies for calculating products and quotients; writing equations to describe quantitative relationships; and relating representations such as equations and diagrams to a situation.

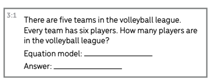
Extending the task

How might students drive the conversation further?

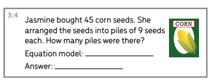
- For each number in an equation model for the situation, such as 4 × □ = 32, students could identify the quantity in the situation to which the number refers. For example, 4 refers to "the number of bulletin boards." Note that naming a quantity is different from naming the numerical value of the quantity.
- Students could check their quotient by multiplying.

Related Math Milestones tasks

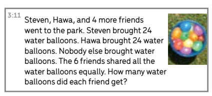
3:1


3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?
Equation model: ________
Answer: ________

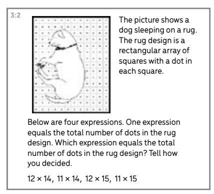
3:4


3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?
Equation model: ________
Answer: ________

3:11


3:11 Steven, Hawa, and 4 more friends went to the park. Steven brought 24 water balloons. Hawa brought 24 water balloons. Nobody else brought water balloons. The 6 friends shared all the water balloons equally. How many water balloons did each friend get?

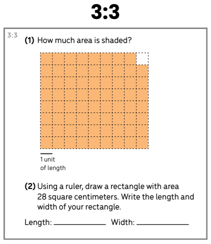
3:2


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

12 × 14, 11 × 14, 12 × 15, 11 × 15

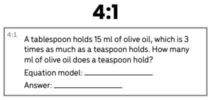
3:3


3:3 **(1)** How much area is shaded?

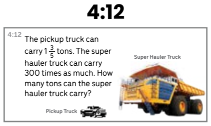
(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.
Length: ________ Width: ________

Task **3:1 Volleyball Players** is a word problem of type *Equal Groups with Product Unknown*; task **3:4 Corn Seeds** is a word problem of type *Equal Groups with Number of Groups Unknown*; and task **3:11 Water Balloons** is a multi-step word problem involving two situation types, one of which is *Equal Groups with Group Size Unknown*. Task **3:2 Hidden Rug Design** involves interpreting a product expression in an array context. Multiplication is useful in task **3:3 Length and Area Quantities**.

4:1


4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?
Equation model: ________
Answer: ________

4:12


4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

In later grades, task **4:1 A Tablespoon of Oil** is a word problem of type *Compare with Size of Smaller Quantity Unknown*, and task **4:12 Super Hauler Truck** is a word problem of type *Compare with Size of Larger Quantity Unknown*.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones](#).

3:9 Bulletin Board Pictures

Teacher Notes

Math Milestones™

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:10 Alice's Multiplication Facts

Teacher Notes

Central math concepts

Parts (1) and (2). The number 7 can be decomposed as 5 + 2; that is, 5 + 2 is another way of writing 7. So, 7 × 8 = (5 + 2) × 8. The distributive property tells us that (5 + 2) × 8 = (5 × 8) + (2 × 8). This is useful if we remember that 8 × 5 = 40 and 8 × 2 = 16, because then we can say that 8 × 7 = 40 + 16. (As was done here, students can be encouraged to use parentheses to remind them that (5+2) is just a number, another way of writing 7.)

The distributive property initially emerges in the curriculum as a consequence of the equal-groups idea of multiplication. However, the distributive property remains applicable not only in grade 3 but also all throughout students' evolving ideas about multiplication, from grade 3 ideas of equal-groups to grades 4 and 5 ideas of times-as-much and scaling, to middle-grades ideas of scaling with signed rational numbers and real numbers, to high school ideas of complex number products. And from the middle grades onward, the distributive property will be the workhorse of algebra with variables and variable expressions. The distributive property is a sturdy principle that stretches across many years of school mathematics, available to be used whenever helpful to connect new ideas with old, to scaffold students' sense-making, to provide representations as a basis for creating and critiquing mathematical arguments, and to extend students' procedural knowledge and fluency with calculations and symbolic manipulations.

Part (3). A way to promote algebraic thinking in arithmetic is to help students view multiplication expressions like 5 × 8 as objects with structure that can be interpreted. An additional way to promote algebraic thinking in arithmetic is to help students to think about statements that refer to infinitely many cases. Students can wonder if what's true for particular numbers might be true for other numbers or all numbers. Part (3) of task 3:10 stretches beyond the particular case of 'the fact that Alice forgot' to aim at an important general principle. The thrust of the problem isn't just that 40 + 16 = 56, or that (5 × 8) + (2 × 8) = 7 × 8, but that there is a principle here that holds true for any numbers. Using letters to stand for 'any number,' this principle is $(b + c) \times a = b \times a + c \times a$.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: multiplying by 2 and by 5; writing multiplication expressions to describe configurations of objects in equal groups; adding two two-digit numbers without regrouping; writing equations to express relationships; and basing multiplicative reasoning and distributive property reasoning on math diagrams.

3:10 Alice forgot what 7 × 8 equals. Alice knows that 5 × 8 = 40 and 2 × 8 = 16.

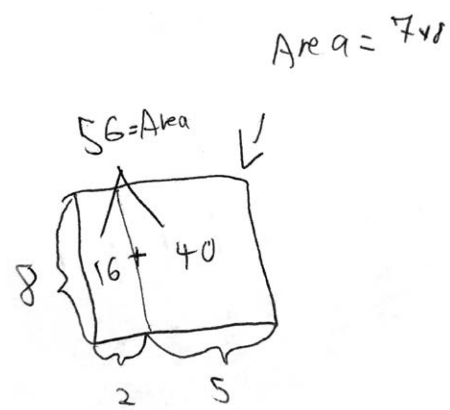
(1) Write a sentence to tell Alice how she can find the value of 7 × 8 by using the two facts she knows.

(2) Draw a diagram that could help Alice understand why your method works.

(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Answer

(1) Answers will vary but should be based on the fact that 7 × 8 = 5 × 8 + 2 × 8. **(2)** Diagrams can vary but could use area, arrays, and/or equal groups to show the relationship between the products 2 × 8, 5 × 8, and 7 × 8 (see example). **(3)** Answers will vary but should involve the idea that products can always be broken into partial products, and/or the idea that a diagram like the student's diagram from part (2) could be drawn for any pair of factors.



Click here for a student-facing version of the task.

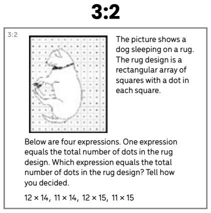
Extending the task

How might students drive the conversation further?

- Students who grasp the underlying distributive property principle at work in the task may still struggle to communicate the principle they understand. A routine such as *Stronger and Clearer Each Time* could be used to refine the explanations students give each other.
- Try all the decompositions of 7 into sums of 2 numbers: 1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, 6 + 1. For example, 7 × 8 = 1 × 8 + 6 × 8 = 8 + 48 = 56. Which decomposition makes the calculation of 7 × 8 easiest for you? Why are the results of all the decompositions the same?
- Students could extend the method to cases involving subtraction, such as finding the value of 4 × 8 by remembering 5 × 8 = 40 and subtracting 8 from 40.
- Students could use the biggest product they know to produce an even bigger product they didn't know before. For example, if a student knows 9 × 9 = 81, then they know that 9 9s are 81, so 18 9s must be 81 + 81; that is, 18 × 9 = 162. (Note that in grade 4, procedures for multiplying multi-digit numbers will use place-value-based decomposition 18 × 9 = 10 × 9 + 8 × 9 over arbitrary decompositions like 18 × 9 = 9 × 9 + 9 × 9, but both decompositions are valid applications of distributivity.)

Related Math Milestones tasks

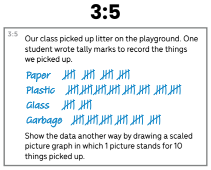
3:2


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

12 × 14, 11 × 14, 12 × 15, 11 × 15

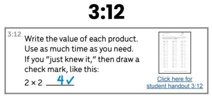
3:5


3:5 Our class picked up litter on the playground. One student wrote tally marks to record the things we picked up.

Paper
Plastic
Glass
Garbage

Show the data another way by drawing a scaled picture graph in which 1 picture stands for 10 things picked up.

3:12

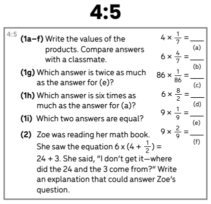

3:12 Write the value of each product. Use as much time as you need. If you "just knew it," then draw a check mark, like this:

2 × 2 4✓

Click here for student handout 3:12

Task **3:2 Hidden Rug Design** involves the equal-groups concept of multiplication and the practice of interpreting the structure of expressions. Task **3:5 Playground Cleanup** is a data representation task that involves equal groups of 5 and 10. Task **3:12 Products of Single-Digit Numbers** is a fluency and recall task for these products.

4:5


4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

(1g) Which answer is twice as much as the answer for (e)?

(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

$4 \times \frac{1}{7} =$ ___ (a)

$6 \times \frac{4}{7} =$ ___ (b)

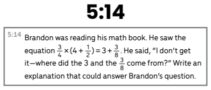
$86 \times \frac{1}{86} =$ ___ (c)

$6 \times \frac{8}{2} =$ ___ (d)

$9 \times \frac{1}{9} =$ ___ (e)

$9 \times \frac{2}{9} =$ ___ (f)

5:14


5:14 Brandon was reading his math book. He saw the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$. He said, "I don't get it—where did the 3 and the $\frac{3}{8}$ come from?" Write an explanation that could answer Brandon's question.

In later grades, task **4:5 Fraction Products and Properties** (part (2)) involves the distributive property as applied to a product of a whole number and a fraction; task **5:14 Brandon's Equation** follows that progression further, featuring a product in which neither factor is a whole number.

Refer to the Standards

3.OA.B.5, 3.MD.C.7b; MP.1, MP.3, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Procedural skill and fluency

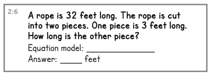
Additional notes on the design of the task

The products 5 × 8 and 2 × 8 are typically easier for students to remember than the product 7 × 8, which makes the strategy of this task pragmatically useful in addition to mathematically important. Calculating an unknown product using known partial products is an important strategy for students during the time when they are still on the path to becoming fluent with and remembering single-digit products. This strategy will also underlie multi-digit multiplication algorithms in later grades (see for example task **4:10 Calculating Products and Quotients**).

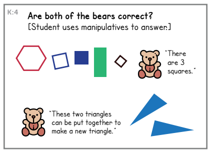
Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:10? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 3:10? In what specific ways do they differ from 3:10?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:10 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

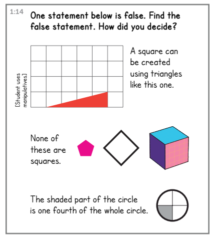
2:6



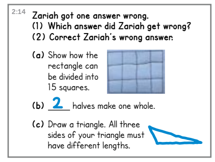
K:4



1:14



2:14



In earlier grades, task **2:6 Cutting a Rope** is a word problem whose equation model consists of a decomposition, 32 = 29 + 3. Spatially, tasks **K:4 Bears Talk About Shapes**, **1:14 Shape True/False**, and **2:14 Correcting a Shape Answer** involve composing and/or decomposing shapes.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:11 Water Balloons

Teacher Notes

Central math concepts

Table 2 of the *Progression* document† lists elementary addition and subtraction situations (p. 9). Table 3 of the document lists elementary multiplication and division situations (p. 23). These elementary situations combine in multi-step problems. Task 3:11 combines an addition situation, *Put Together/Take Apart with Total Unknown*, and a multiplication/division situation, *Equal Groups of Objects with Group Size Unknown*.

Equations are an important way to represent the relationships between quantities in a situation. In task 3:11, the equations

$24 + 24 = 48$

$48 \div 6 = 8$

could be seen as "telling the story" of the problem in the following way. First, the equation $24 + 24 = 48$ describes a process of Steven and Hawa putting their balloons together. Then, the equation $48 \div 6 = 8$ describes a process of 6 friends sharing those balloons equally.

Another way to tell the story could be with the following equations:

$24 \div 6 = 4$

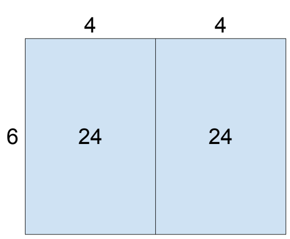
$24 \div 6 = 4$

$4 + 4 = 8.$

In this version of the story, one of the friends first receives a share of Steven's balloons (as represented in the first equation $24 \div 6 = 4$) and next receives a share of Hawa's balloons (as represented in the second equation $24 \div 6 = 4$). Then the equation $4 + 4 = 8$ shows those two shares being combined.

It is intuitively clear that the situation on the playground could have played out in either way, but how does it happen mathematically that the final answer comes out the same regardless of which of these two approaches we choose? The guarantee in this case is the distributive property:

$$(24 + 24) \div 6 = 24 \div 6 + 24 \div 6.$$



Relating an equation to the situation it describes involves viewing an expression like $24 + 24$ not only as "a step" toward a result of 48, but also as an object that can be interpreted before evaluating it, based on the meaning of the operation and the role of the numbers in the situation. $24 + 24$ equals 48, to be sure, but $24 + 24$ left unevaluated is also a valuable record of an operation of *putting two collections together*. This

3:11 Steven, Hawa, and 4 more friends went to the park. Steven brought 24 water balloons. Hawa brought 24 water balloons. Nobody else brought water balloons. The 6 friends shared all the water balloons equally. How many water balloons did each friend get?

Answer

Each friend got 8 water balloons.

Click here for a student-facing version of the task.

Refer to the Standards

3.OA.D.8; MP.1, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

The number 4 in the first sentence likely doesn't enter into the solution of the problem, unless it is used to determine the total number of friends. This value is stated explicitly later in the text of the problem.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:11? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 3:11? In what specific ways do they differ from 3:11?

perspective on a numerical expression expands students' perspective on arithmetic, prefiguring their work with algebraic expressions in the middle grades (CCSS 6.EE.A.2). As observed in the *Progression* document,

> [Students] understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. (p. 13)

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: adding two two-digit numbers; using addition in context; finding single-digit products and related quotients; using multiplication and division in context; and writing situation equations and solution equations.

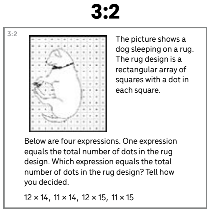
Extending the task

How might students drive the conversation further?

- Students could be asked how many water balloons each friend would get if a third friend had brought 6 water balloons to add to the total.
- For that version of the problem, students could compare and discuss two methods, such as (a) 48 + 6 = 54, 54 ÷ 6 = 9; (b) 8 + 1 = 9 because 6 more balloons provides 1 more balloon for each friend.

Related Math Milestones tasks

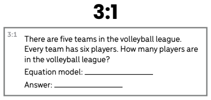
3:2


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

12 × 14, 11 × 14, 12 × 15, 11 × 15

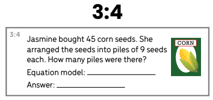
3:1


3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?

Equation model: ____________

Answer: ____________

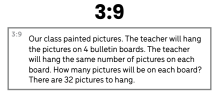
3:4


3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?

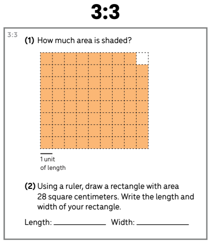
Equation model: ____________

Answer: ____________

3:9


3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

3:3


3:3 (1) How much area is shaded?

(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ____________ Width: ____________

Task **3:2 Hidden Rug Design** centers on the equal-groups concept of multiplication in an array context, in a way that involves viewing expressions as objects with structure. Tasks **3:1 Volleyball Players**, **3:4 Corn Seeds**, and **3:9 Bulletin Board Pictures** are word problems centered on equal-groups concepts of multiplication and/or division. Multiplication is useful in task **3:3 Length and Area Quantities**.

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:11 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2011, May 29). *Progressions for the Common Core State Standards in Mathematics (draft) K, Counting and Cardinality; K–5, Operations and Algebraic Thinking*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

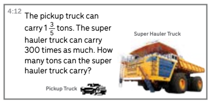
4:1

4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ____________

Answer: ____________

4:12

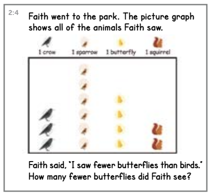


5:13

5:13 In a snack shop there is a frozen yogurt machine. When there is 3 *l* of frozen yogurt in the machine, the machine is $\frac{1}{3}$ full. How much frozen yogurt is in the machine when it is $\frac{1}{4}$ full?

In later grades, tasks **4:1 A Tablespoon of Oil** and **4:12 Super Hauler Truck** are word problems involving multiplicative comparison (see Table 3, p. 23 of the relevant *Progression* document). Task **5:13 Frozen Yogurt Machine** combines an unknown-factor situation with an unknown-product situation, in a context involving fractional quantities.

2:4



In earlier grades, task **2:4 Animals in the Park** combines the situation types *Put Together/Take Apart with Total Unknown and Compare with Difference Unknown ('how many fewer' language)* (see Table 2, p. 9 of the relevant *Progression* document).

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:12 Products of Single-Digit Numbers

Teacher Notes

Central math concepts

Task 3:12 draws on memory and fluency. The table illustrates how the 66 brief problems in task 3:12 map to the multiplication table.

×	0	1	2	3	4	5	6	7	8	9
0										
1		✔						✔		
2				✔	✔	✔	✔	✔	✔	✔
3			✔	✔	✔	✔	✔	✔	✔	✔
4		✔	✔	✔	✔	✔	✔	✔	✔	✔
5			✔	✔	✔	✔	✔	✔	✔	✔
6			✔	✔	✔	✔	✔	✔	✔	✔
7			✔	✔	✔	✔	✔	✔	✔	✔
8			✔	✔	✔	✔	✔	✔	✔	✔
9			✔	✔	✔	✔	✔	✔	✔	✔

The values of the single-digit products play a large role in school mathematics during the elementary grades, the middle grades, and high school. For example, the problem of calculating a multi-digit product can be reduced to summing a series of terms that are single-digit products multiplied by powers of ten (see sidebar). The standard pencil-and-paper algorithm for multi-digit multiplication is an efficient bookkeeping method for this process.

Rewriting the product 734 × 8 as a sum of single-digit products multiplied by powers of ten:

$734 \times 8 = (700 + 30 + 4) \times 8$

$= 700 \times 8 + 30 \times 8 + 4 \times 8$

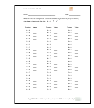
$= (\mathbf{7 \times 8}) \times 100 + (\mathbf{3 \times 8}) \times 10 + (\mathbf{4 \times 8})$.

Indeed, many mathematical tasks in grade 3 and beyond are facilitated by remembering single-digit products and being fluent with related quotients. Examples include:

- Finding final answers to word problems in multiplication and division situations;
- Calculating multi-digit products and quotients, and assessing the results of such calculations by estimating;
- Understanding, recognizing, and generating equivalent fractions;
- Reasoning about products and quotients of fractions;
- Factoring composite numbers and multiplying out prime factorizations;
- Understanding, recognizing, and generating equivalent ratios, and seeing patterns in ratio tables;
- Reasoning with ratios and seeing patterns in ratio tables;

3:12 Write the value of each product. Use as much time as you need. If you "just knew it," then draw a check mark, like this:

2 × 2 4✓



Click here for student handout 3:12

Answer

Click here for an answer key. Students' check marks may vary by individual.

Click here for a student-facing version of the task.

Refer to the Standards

3.OA.C.7; MP.1, MP.6. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

- The task is designed to be worked on after all the products in the multiplication table have been understood and practiced.
- The instructions say, "Use as much time as you need." Reasons for this include: (1) Differentiating between students on the basis of their speed isn't the purpose of the task. (2) More generally, speed isn't an important disciplinary value in mathematics. (3) Emphasizing speed in the mathematical community of the classroom can have negative effects on students' mathematics identity.

- Solving problems involving unit rates, linear functions, percents, unit conversions, similar figures, and other instances of scaling and proportionality;
- Factoring quadratic expressions and other polynomial expressions; and
- Counting possible outcomes to determine probabilities.

Remembering single-digit products and being fluent with related quotients is therefore an important goal (CCSS 3.OA.C.7). This goal needs to be reached by an intellectually valid, emotionally supportive learning path. The mathematical stages of that path are articulated in the *Progression* document, under the heading "Levels in problem representation and solution" (see pp. 25–27).†

As noted also in the *Progression* document (p. 27), "Organizing practice so that it focuses most heavily on understood but not yet fluent products and unknown factors can speed learning. To achieve this by the end of Grade 3, students must begin working toward fluency for the easy numbers as early as possible. Because an unknown factor (a division) can be found from the related multiplication, the emphasis at the end of the year is on knowing from memory all products of two one-digit numbers. … [T]his isn't a matter of instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involves the interplay of practice and reasoning."‡ Also note (p. 22) that "mastering this material, and reaching fluency in single-digit multiplications and related divisions with understanding, may be quite time consuming …. So it is imperative that extra time and support be provided if needed." In addition, grades 3 and 4 teachers could co-develop a plan for extending and/or maintaining recall and fluency with single-digit products and related quotients, as needed, during grade 4.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: Level 3 strategies as described in the *Progression* document under the heading "Levels in problem representation and solution" (see pp. 25–27).

Extending the task

How might students drive the conversation further?

- Students could notice that a certain factor is absent from the student handout, or be asked which factor is absent. What might be the reason why no products with this factor were included?
- Students could circle several products they feel they could use more practice with.

Additional notes on the design of the task (continued)

- The instructions say, "If you 'just knew it,' then draw a check mark." This is intended to provide information about which single-digit products are known from memory.
- The task includes relatively few products that involve 1 as a factor, and it includes no products that involve 0 as a factor. All such products are instances of the general patterns $1 \times n = n \times 1 = 1$ and $0 \times n = n \times 0 = 0$. (See "Extending the task.")

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:12? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 3:12? In what specific ways do they differ from 3:12?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:12 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2011, May 29). *Progressions for the Common Core State Standards in Mathematics (draft) K, Counting and Cardinality; K–5, Operations and Algebraic Thinking.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ For further discussion of such an interwoven process, see "By the end of grade 3: Developing fluency with multiplication and division" (blog post by Hill, 2021).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Related Math Milestones tasks

3:13

3:13 Write the number that makes each equation true. Use as much time as you need.

Click here for student handout 3:13

3:1

3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?

Equation model: ______________

Answer: ______________

3:4

3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?

Equation model: ______________

Answer: ______________

3:9

3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

3:11

3:11 Steven, Hawa, and 4 more friends went to the park. Steven brought 24 water balloons. Hawa brought 24 water balloons. Nobody else brought water balloons. The 6 friends shared all the water balloons equally. How many water balloons did each friend get?

3:10

3:10 Alice forgot what 7 × 8 equals. Alice knows that 5 × 8 = 40 and 2 × 8 = 16.

(1) Write a sentence to tell Alice how she can find the value of 7 × 8 by using the two facts she knows.

(2) Draw a diagram that could help Alice understand why your method works.

(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Task **3:13 Fluency within the Multiplication Table** includes problems about fact families, such as 21 ÷ 7 = ☐, ☐ × 8 = 16, ☐ ÷ 3 = 5, and 12 ÷ ☐ = 2, in which an unknown number is sought that makes an equation true. Single-digit products and related quotients are involved in the word problems in tasks **3:1 Volleyball Players**, **3:4 Corn Seeds**, **3:9 Bulletin Board Pictures**, and **3:11 Water Balloons**. Task **3:10 Alice's Multiplication Fact** involves a distributive property strategy for using known products to determine an unknown product.

4:10

4:10 Write the values of the products and quotients. Check the quotients by multiplying.

Mentally: 40 × 20, 30 × 11, 12 × 60, 5 × 19, 480 ÷ 8

With pencil and paper: 6,132 × 6, 48 × 39, 7)8,722

5:5

5:5 Write the requested values.

4087 × 53 = ?	$\frac{1}{10} \div 10 = ?$	0.4 × 0.9 = ?
246 × 914 = ?	$\frac{7}{8} \times \frac{5}{3} = ?$	0.75 ÷ 0.01 = ?
9744 ÷ 12 = ?	8 × ? = 73	0.63 ÷ 0.3 = ?
1461 ÷ 6 = ?	$3 \div \frac{1}{8} = ?$	0.86 + 0.4 = ?
4 − (8 − 4) = ?	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	0.72 − 0.17 = ?
	$\frac{1}{3} \div (6 \times 5) = ?$	0.02 + 0.2 = ?
		0.8 − 0.55 = ?
		637 − 1.31 = ?

6:14

6:14 *Pencil and paper* **(1)** 81.53 ÷ 3.1 = ? **(2)** $\frac{7}{8} \div \frac{2}{3} = ?$ **(3)** Check both of your answers by multiplying.

In later grades, task **4:10 Calculating Products and Quotients** involves grade-level procedures with multi-digit multiplication and division. Task **5:5 Calculating** continues procedures into larger numbers of digits and into fractions and decimals. Task **6:14 Dividing Decimals and Fractions** marks the culmination of algorithmic procedures with general fraction and decimal division.

2:5

2:5 Write the value of each sum. Use as much time as you need. If you "just knew it," then draw a check mark, like this: 2 + 2 4✓

Click here for student handout 2:5

In earlier grades, task **2:5 Sums of Single-Digit Numbers** is the analogue of task 3:12 for addition.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:13 Fluency within the Multiplication Table

Teacher Notes

Central math concepts

Task 3:13 draws on memory, fluency, and conceptual understanding. In terms of conceptual understanding, the central mathematical idea in task 3:13 is that $C \div A$ is the unknown factor in $A \times ? = C$. This idea expresses the mathematical relationship between multiplication and division, whether for whole numbers, fractions, decimals, variables, variable expressions, or complex numbers. The 66 brief problems in task 3:13 involve different permutations of this relationship, as shown in the table.

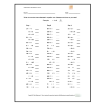
Example Equation	Equation Type	How many are in task 3:13?
21 ÷ 7 = □	Unknown Quotient	34
□ × 8 = 16 9 × □ = 45	Unknown Factor	18
□ ÷ 3 = 5	Unknown Dividend	7
12 ÷ □ = 2	Unknown Divisor	7

Task 3:13 doesn't include equations of type Unknown Product (for example, 3 × 8 = □), because products like 3 × 8 are the topic of task **3:12 Products of Single-Digit Numbers**. Whereas the problems in that task simply ask for the value of an expression like 3 × 8, the problems in task 3:13 ask for an unknown number that makes an equation true.

Many mathematical tasks in grade 3 and beyond are facilitated by remembering single-digit products and being fluent with related quotients:

- Finding final answers to word problems in multiplication and division situations;
- Calculating multi-digit products and quotients, and assessing the results of such calculations by estimating;
- Understanding, recognizing, and generating equivalent fractions;
- Reasoning about products and quotients of fractions;
- Factoring composite numbers and multiplying out prime factorizations;
- Understanding, recognizing, and generating equivalent ratios, and seeing patterns in ratio tables;
- Reasoning with ratios and seeing patterns in ratio tables;
- Solving problems involving unit rates, linear functions, percents, unit conversions, similar figures, and other instances of scaling and proportionality;

3:13 Write the number that makes each equation true. Use as much time as you need.



Click here for student handout 3:13

Answer

Click here for an answer key.

Click here for a student-facing version of the task.

Refer to the Standards

3.OA.C.7, 3.OA.A.4, 3.OA.A, B, C; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

- The task is designed to be worked on after all the products and related quotients in the multiplication table have been understood and practiced.
- The instructions say, "Use as much time as you need." Reasons for this include: (1) Differentiating between students on the basis of their speed isn't the purpose of the task. (2) More generally, speed isn't an important disciplinary value in mathematics. (3) Emphasizing speed in the mathematical community of the classroom can have negative effects on students' mathematics identity.

- Factoring quadratic expressions and other polynomial expressions; and
- Counting possible outcomes to determine probabilities.

Remembering single-digit products and being fluent with related quotients is therefore an important goal (CCSS 3.OA.C.7). This goal needs to be reached by an intellectually valid, emotionally supportive learning path. The mathematical stages of that path are articulated in the *Progression* document, under the heading "Levels in problem representation and solution" (see pp. 25–27).[†]

As noted also in the *Progression* document (p. 27), "Organizing practice so that it focuses most heavily on understood but not yet fluent products and unknown factors can speed learning. To achieve this by the end of Grade 3, students must begin working toward fluency for the easy numbers as early as possible. Because an unknown factor (a division) can be found from the related multiplication, the emphasis at the end of the year is on knowing from memory all products of two one-digit numbers. … [T]his isn't a matter of instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involves the interplay of practice and reasoning."[‡] Also note (p. 22) that "mastering this material, and reaching fluency in single-digit multiplications and related divisions with understanding, may be quite time consuming …. So it is imperative that extra time and support be provided if needed." In addition, grades 3 and 4 teachers could co-develop a plan for extending and/or maintaining recall and fluency with single-digit products and related quotients, as needed, during grade 4.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: the relationship between multiplication and division; and remembered single-digit products.

Extending the task

How might students drive the conversation further?

- Checking selected equations with division signs by multiplying can offer additional procedural practice and reinforce the relationship between multiplication and division ($C \div A$ is the unknown factor in $A \times \square = C$).
- Similarly, given selected completed equations (such as $15 \div 3 = 5$), students could write an equivalent equation (such as $3 \times 5 = 15$ or $15 \div 5 = 3$).

Additional notes on the design of the task (continued)

- The task has a compensatory design that balances difficulty along three dimensions.
  - *Balancing the equation types with the number of problems to solve*. As shown in the table of equation types (see "Central math concepts"), the number of tasks of each equation type decreases as the likely difficulty of the equation type increases
  - *Balancing the equation types with the fact families*. For example, compare $54 \div 9 = \square$ with $12 \div \square = 2$. The likely easier equation type (Unknown Quotient) has been posed with the likely more difficult fact family ($9 \times 6 = 54$), whereas the likely more difficult equation type (Unknown Divisor) has been posed with the easier fact family ($6 \times 2 = 12$).
  - *Balancing the problems with the days*. Equations on Day 1 are intended to be generally easier than those on Day 2. Day 3 ends with property-based problems involving factors of 1 and 0.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:13? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 3:13? In what specific ways do they differ from 3:13?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:13 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

Related Math Milestones tasks

3:12

3:12 Write the value of each product. Use as much time as you need. If you "just knew it," then draw a check mark, like this:
2×2 4 ✓
Click here for student handout 3:12

3:1

3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?
Equation model: ____________
Answer: ____________

3:4

3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?


Equation model: ____________
Answer: ____________

3:9

3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

3:11

3:11 Steven, Hawa, and 4 more friends went to the park. Steven brought 24 water balloons. Hawa brought 24 water balloons. Nobody else brought water balloons. The 6 friends shared all the water balloons equally. How many water balloons did each friend get?


3:10

3:10 Alice forgot what 7×8 equals. Alice knows that $5 \times 8 = 40$ and $2 \times 8 = 16$.
(1) Write a sentence to tell Alice how she can find the value of 7×8 by using the two facts she knows.
(2) Draw a diagram that could help Alice understand why your method works.
(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Task **3:12 Products of Single-Digit Numbers** asks for the values of products in the multiplication table. Single-digit products and related quotients are involved in the word problems in tasks **3:1 Volleyball Players**, **3:4 Corn Seeds**, **3:9 Bulletin Board Pictures**, and **3:11 Water Balloons**. Task **3:10 Alice's Multiplication Fact** involves a distributive property strategy for using known products to determine an unknown product.

4:10

4:10 Write the values of the products and quotients. Check the quotients by multiplying.

Mentally: 40×20, 30×11, 12×60, 5×19, $480 \div 8$

With pencil and paper: $6{,}132 \times 6$, 48×39, $8{,}722 \div 7$

5:5

5:5 Write the requested values.

$4087 \times 53 = ?$	$\frac{1}{10} \div 10 = ?$	$0.4 \times 0.9 = ?$
$246 \times 914 = ?$	$\frac{7}{8} \times \frac{5}{3} = ?$	$0.75 \div 0.01 = ?$
$9744 \div 12 = ?$	$8 \times ? = 73$	$0.63 \div 0.3 = ?$
$1461 \div 6 = ?$	$3 \div \frac{1}{8} = ?$	$0.86 + 0.4 = ?$
$4 - (8 - 4) = ?$	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	$0.72 - 0.17 = ?$
	$\frac{1}{3} \div (6 \times 5) = ?$	$0.02 + 0.2 = ?$
		$0.8 - 0.55 = ?$
		$637 - 1.31 = ?$

6:14

6:14 *Pencil and paper* **(1)** $81.53 \div 3.1 = ?$ **(2)** $\frac{7}{8} \div \frac{2}{3} = ?$ **(3)** Check both of your answers by multiplying.

In later grades, task **4:10 Calculating Products and Quotients** involves grade-level procedures with multi-digit multiplication and division; these procedures are built upon single-digit calculations as in task 3:13. Task **5:5 Calculating** continues procedures into larger numbers of digits and into fractions and decimals. Task **6:14 Dividing Decimals and Fractions** marks the culmination of algorithmic procedures with general fraction and decimal division.

2:3

2:3 Write the sums and differences.

$\begin{array}{r} 36 \\ +45 \\ \hline \end{array}$ $\begin{array}{r} 72 \\ -17 \\ \hline \end{array}$ $\begin{array}{r} 64 \\ +27 \\ \hline \end{array}$ $\begin{array}{r} 82 \\ -55 \\ \hline \end{array}$

In earlier grades, task **2:3 Fluency within 100 (Add/Subtract)** is the analogue of task 3:13 for addition and subtraction.

† Common Core Standards Writing Team. (2011, May 29). *Progressions for the Common Core State Standards in Mathematics (draft), K, Counting and Cardinality; K–5, Operations and Algebraic Thinking.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ For further discussion of such an interwoven process, see "By the end of grade 3: Developing fluency with multiplication and division" (blog post by Hill, 2021).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

3:14 Fluency within 1000 (Add/Subtract)

Teacher Notes

Central math concepts

Task 3:14 focuses on fluency with procedures. For the mental calculations in the task, students can use place value, properties of operations, and the relationship between addition and subtraction as computation strategies. For the pencil-and-paper calculations in the task, there are various possibilities for efficient, accurate, and generalizable strategies and algorithms that handle the given problems.

Computation strategies and computation algorithms are usefully distinguished (CCSS, pp. 85; see figure). Strategies are "purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another." Mental calculation often uses such strategies. For example, we could begin calculating 612 - 13 mentally by thinking that the answer will be 1 less than 612 - 12, because subtracting 12 "takes away 1 too few" compared to the given problem. Alternatively, we could think of 612 - 13 as 1 less than 613 - 13, because "we added 1 compared to the given problem." Flexible procedural fluency based on place value, properties of operations, relationships between operations, and ideas of equivalence is valuable both in practical terms and also as a way of doing arithmetic that prefigures algebra. In algebra, strategies are the most common way of working.

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Algorithms are inflexible by definition. One step follows another in the prescribed order. Some algorithms are much simpler to execute than others. The standard multi-digit addition algorithm is less complex than the standard multi-digit subtraction algorithm. The standard algorithm for dividing fractions is less complex than the standard algorithm for dividing multi-digit numbers.

Sometimes even an efficient general-purpose algorithm wouldn't be an efficient approach to a particular instance of a calculation, as in the addition problem 999 + 1. On the other hand, when faced with a calculation there may be times when we don't find ourselves readily inventing a flexible mental procedure on the spot, so it is valuable to know and be proficient with an algorithm.

The standard multi-digit addition algorithm and the standard multi-digit subtraction algorithm aren't required in grade 3 (CCSS 3.NBT.A.2), although students are expected to use algorithms for addition and subtraction (not just use opportunistic methods), and the standard algorithms can be among those students learn to use. Practice with calculating multi-digit sums and differences can promote confidence while offering opportunities to debug procedures, reinforce ideas, and strengthen recall of single-digit sums and related differences.

3:14 Write the sums and differences.

With pencil and paper

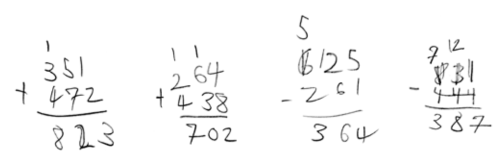
351	264	625	831
+ 472	+ 438	− 261	− 444

Mentally

800 − 300
240 + 540
365 − 165
612 − 13

Answer

For the written calculations, 823, 702, 364, and 387. (See the first figure; students might use different strategies and/or algorithms than the ones shown in the example.) For the mental calculations, 500, 780, 200, and 599. (Some potential thought processes leading to the results—not the only possible thought processes—are suggested in the second figure. Note that for the mental calculations, students could manage the mental load by using writing to jot down intermediate results along the way.)



800 - 300 = **(8 - 3) × 100 = 500**

240 + 540 = **(200 + 500) + (40 + 40) = 700 + 80 = 780**

365 - 165 = **300 - 100 + 0 = 200**

612 - 13 = **613 - 13 - 1 = 600 - 1 = 599**

Click here for a student-facing version of the task.

Refer to the Standards

3.NBT.A.2; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value concepts; and single-digit sums and differences.

Extending the task

How might students drive the conversation further?

- Checking differences by adding can offer additional procedural practice and reinforce the relationship between addition and subtraction (C - A is the unknown factor in A + □ = C).
- Students could make sense of their answers another way by making estimates of the values; for example, the difference 831 - 444 should be reasonably close to 800 - 400 = 400.

Related Math Milestones tasks

3:10

3:10 Alice forgot what 7 × 8 equals. Alice knows that 5 × 8 = 40 and 2 × 8 = 16.
(1) Write a sentence to tell Alice how she can find the value of 7 × 8 by using the two facts she knows.
(2) Draw a diagram that could help Alice understand why your method works.
(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Fluency with two-digit sums and differences is an ingredient in the partial-products (distributive property) relationships in task **3:10 Alice's Multiplication Fact**.

4:14

4:14 540,909 + 87,808 – 5,864 + 2,556 = ?

4:10

4:10 Write the values of the products and quotients. Check the quotients by multiplying.

Mentally: 40 × 20, 30 × 11, 12 × 60, 5 × 19, 480 ÷ 8

With pencil and paper: $6{,}132 \times 6$, 48×39, $8{,}722 \div 7$

5:5

5:5 Write the requested values.

4087 × 53 = ?
246 × 914 = ?
9744 ÷ 12 = ?
1461 ÷ 6 = ?
4 – (8 – 4) = ?
$\frac{1}{10} \div 10 = ?$
$\frac{7}{8} \times \frac{5}{3} = ?$
8 × ? = 73
$3 \div \frac{1}{8} = ?$
$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$
$\frac{1}{3} \div (6 \times 5) = ?$
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637 – 1.31 = ?

6:14

6:14 *Pencil and paper* **(1)** 81.53 ÷ 3.1 = ? **(2)** $\frac{7}{8} \div \frac{2}{3} = ?$ **(3)** Check both of your answers by multiplying.

In later grades, task **4:14 Fluency with Multi-Digit Sums and Differences** marks the culmination of the progression for multi-digit addition and subtraction. Task **4:10 Calculating Products and Quotients** involves grade-level procedures with multi-digit multiplication and division. Task **5:5 Calculating** continues procedures into larger numbers of digits and into fractions and decimals. Task **6:14 Dividing Decimals and Fractions** marks the culmination of algorithmic procedures with general fraction and decimal division.

2:3

2:3 Write the sums and differences. $36 + 45$, $72 - 17$, $64 + 27$, $82 - 55$

2:5

2:5 Write the value of each sum. Use as much time as you need. If you "just knew it," then draw a check mark, like this: 2 + 2 4✓

Click here for student handout 2:5

2:8

2:8 Write the number that makes each equation true. Use as much time as you need.

Click here for student handout 2:8

In earlier grades, task **2:3 Fluency within 100 (Add/Subtract)** is the analogue of task 3:14 for two-digit sums and differences. Tasks **2:5 Sums of Single-Digit Numbers** and **2:8 Fluency within the Addition Table** involve the single-digit sums and related differences upon which multi-digit addition and subtraction algorithms are built.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

The task does not require students to show their work, but looking at students' steps can show where they may have made a careless mistake.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 3:14? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 3:14? In what specific ways do they differ from 3:14?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 3:14 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

3:14 Fluency within 1000 (Add/Subtract)

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
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Language

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