# 4:5 Fraction Products and Properties

Teacher Notes

Math Milestones™

## Central math concepts

**Parts (1a–f)**. Extending multiplication and division from whole numbers to fractions may be the most important content progression of grades 3-5. This progression traces a substantial evolution in students' concepts about numbers and operations. The need to support students in making that conceptual evolution raises important questions: What aspects of earlier thinking about multiplying whole numbers will remain helpful when making sense of a product involving fractions? What new ways of thinking will be helpful? And what mathematical representations introduced during whole-number multiplication work are best suited to supporting the transition from multiplying and dividing whole numbers to multiplying and dividing all numbers?

An early step in the progression of multiplication and division operations is making sense of multiplying a fraction by a whole number. The key ideas involved are listed in the table. An important aspect of working with these ideas is viewing expressions like $86 \times \frac{1}{86}$ not just as instructions to calculate a resulting value, but also as objects with structure that can be interpreted while delaying that evaluation.

**Key Ideas Leading to Multiplying Fractions by Whole Numbers.**

**I. The unit fraction idea.** (Example: $\frac{1}{5}$)

- $\frac{1}{5}$ mile means 1 part of a partition of 1 mile into 5 equal parts.
- 5 parts of size $\frac{1}{5}$ make 1 mile, that is, $5 \times \frac{1}{5} = 1$.
- Expressing the reasoning more generally leads to $B \times \frac{1}{B} = 1$.

**II. Building general fractions from unit fractions.** (Example: $2 \times \frac{1}{5}$)

- $\frac{2}{5}$ mile means 2 parts of size $\frac{1}{5}$ mile, or twice as much as $\frac{1}{5}$ mile.
- That is to say, $\frac{2}{5}$ means $2 \times \frac{1}{5}$.
- In terms of unit thinking, $\frac{2}{5}$ is 2 fifths, or 2 units of one-fifth.
- Expressing the reasoning more generally, $\frac{A}{B}$ means $A \times \frac{1}{B}$ and $A \times \frac{1}{B} = \frac{A}{B}$.

**III. Multiplying general fractions by whole numbers.** (Example: $4 \times \frac{3}{5}$)

- $\frac{3}{5}$ means 3 parts of size $\frac{1}{5}$.
- 4 times as much as 3 parts of size $\frac{1}{5}$ amounts to 4 × 3 parts of size $\frac{1}{5}$.
- 4 × 3 parts of size $\frac{1}{5}$ equal the fraction $\frac{4 \times 3}{5}$. (By the principles of Row II.)
- In terms of unit thinking, 4 times (3 fifths) = (4 times 3) fifths. This is like the problem 4 × 30 = 4 × (3 tens) = (4 × 3) tens. The units of tens are analogous to the units of fifths. And the fundamental principle at work is the associative property of multiplication, $r \times (s \times t) = (r \times s) \times t$.
- Expressing the reasoning more generally, $A \times \frac{C}{D} = \frac{A \times C}{D}$.

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} =$ ___ (a)

$6 \times \frac{4}{7} =$ ___ (b)

$86 \times \frac{1}{86} =$ ___ (c)

$6 \times \frac{8}{2} =$ ___ (d)

$9 \times \frac{1}{9} =$ ___ (e)

$9 \times \frac{2}{9} =$ ___ (f)

**(1g)** Which answer is twice as much as the answer for (e)?

**(1h)** Which answer is six times as much as the answer for (a)?

**(1i)** Which two answers are equal?

**(2)** Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

### Answer

**(1)** (a) $\frac{4}{7}$. (b) $\frac{24}{7}$ or $3\frac{3}{7}$. (c) 1. (d) $\frac{48}{2}$, $\frac{24}{1}$, or 24. (e) 1. (f) $\frac{18}{9}$, $\frac{6}{3}$, $\frac{2}{1}$, or 2. (g) $\frac{18}{9}$, $\frac{6}{3}$, $\frac{2}{1}$, or 2. (h) $\frac{24}{7}$ or $3\frac{3}{7}$. (i) 1 and 1 (from parts (c) and (e)). **(2)** Answers will vary. One kind of explanation involves calculating $6 \times 4\frac{1}{2} = 6 \times \frac{9}{2} = \frac{54}{2} = 27$, then recognizing that 27 = 24 + 3.

A second kind of explanation involves an application of the distributive property: Because 6 × 4 = 24 and $6 \times \frac{1}{2} = 3$, we can multiply $6 \times (4 + \frac{1}{2})$ by adding the products: $6 \times (4 + \frac{1}{2}) = 6 \times 4 + 6 \times \frac{1}{2} = 24 + 3$. This reveals 24 and 3 as partial products. A third kind of explanation may take advantage of the bidirectionality of the equal sign by beginning the reasoning with 24 + 3 and working from right to left. (For example, 24 + 3 equals 3 times 8 + 1, and 8 + 1 is twice as much as $4 + \frac{1}{2}$, so altogether 24 + 3 equals 3 × 2 times $4 + \frac{1}{2}$.)

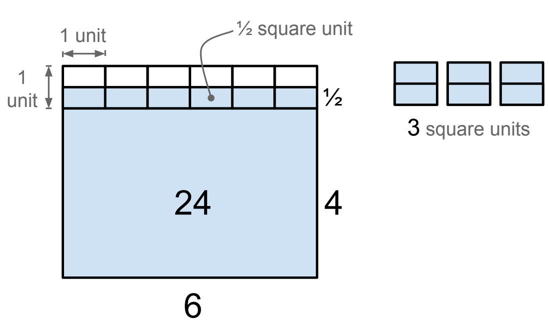
**Part (2)**. One way in which we often use the distributive property, explicitly or implicitly, is to help us calculate a product of sums. For elementary-grades students, the first such problems arise when calculating products of multi-digit numbers; this is because every multi-digit number is a sum of terms. For example, 19 is a sum 10 + 9, which means we could calculate 5 × 19 mentally by thinking "It's 50 + 45," or "It's 100 - 5."

Upper-elementary students next apply the distributive property to products and sums that involve fractions. That development or progress is important because in middle grades, students will apply the distributive property to products and sums that involve not only whole numbers and fractions, but also rational numbers, real numbers, variables, and variable expressions. High school students will apply the distributive property not only to real numbers and variable expressions, but also to complex numbers, and perhaps to matrices. If the inner structure of arithmetic and algebra could be likened to a skeleton, then the distributive property would be the backbone.

In the symbol 378, there's nothing visible that tells us it refers to a sum; primary-grades students must learn to unpack 378 as 300 + 70 + 8. Similarly, upper-elementary students learn to unpack a symbol like 7.3 as $7 + \frac{3}{10}$. Mixed numbers are also sums, although this is not indicated by a symbol like $4\frac{1}{2}$. Fortunately, the symbol $4\frac{1}{2}$ is read aloud as "Four *and* one-half," where the word *and* supports correct interpretation of the symbol. Still, even high school students can sometimes accidentally misinterpret a symbol like $4\frac{1}{2}$ or have difficulty entering the number into a calculator. Their fluency in the conventions of algebra (in particular, the use of juxtaposition to indicate multiplication) can occasionally mislead them into thinking that $4\frac{1}{2}$ refers to the product $4 \times \frac{1}{2}$ rather than the sum $4 + \frac{1}{2}$.

A student who efficiently calculates $6 \times 4\frac{1}{2} = 6 \times \frac{9}{2} = \frac{54}{2} = 27$ has demonstrated a valuable fluency. Efficiently evaluating fraction products is important. Also important, especially as preparation for algebra, is being able to study expressions like the ones on either side of the equal sign in Zoe's equation, and *delay* evaluation of them in order to analyze the expression as an object with structure.

Thus, the most insightful explanation of the equation $6 \times 4\frac{1}{2} = 24 + 3$ arguably isn't simply to observe that the numerical values on either side of the equal sign are equal. For example, the same area models that illustrate the distributive property for whole numbers could be used to make sense of the numbers in Zoe's equation as partial products:




### Answer (continued)

Answers may include such explanatory techniques as, for example: showing a math diagram, such as an area model; creating a simple word problem that makes sense of the numbers in Zoe's equation; and/or writing expressions and equations.

Click here for a student-facing version of the task.

### Refer to the Standards

4.NF.B.4a, 4b; MP.3, MP.5, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

### Aspect(s) of rigor:

Concepts

### Additional notes on the design of the task

- Questions (1a–f) can be answered solely using procedural knowledge. The intent of the follow-up questions (1g–i) is to invite students to notice patterns in the results and make sense of the products, for example by relating the size of the product to the size of the factors.
- Zoe's question specifically asks, "Where did the 24 and the 3 come from?"—not just "I don't get it," or "Is this equation true?" The phrasing is intended to invite sense-making about the equation. Implicit in Zoe's question, perhaps, is that the confusing numbers 24 and 3 are addends; so part of Zoe's confusion might be that a calculation like $6 \times 4\frac{1}{2}$, in which we were supposed to multiply, got turned into an addition problem.

To be sure, it is mathematically valid to explain Zoe's equation by observing that the numerical values on either side of the equal sign are equal—it's an ironclad proof. That explanation also demonstrates an understanding of the meaning of the equal sign. Thus, the purpose of task 4:5 isn't to differentiate between the students who explain things one way and the students who explain things the other way; the purpose is to bring different explanations together, and relate those explanations to one another, so that all students deepen their understanding and build a foundation for future learning.

## Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: recall of single-digit products; multiplying $\frac{1}{2}$ by an even number; and basing multiplicative reasoning and distributive property reasoning on math diagrams.

## Extending the task

How might students drive the conversation further?

- Building on the particular cases $9 \times \frac{1}{9} = 1$ and $86 \times \frac{1}{86} = 1$, students could propose, and create a justification for, a statement that generalizes from those cases, either verbally or by creating a mathematical statement such as "$B \times \frac{1}{B} = 1$ for any unit fraction $\frac{1}{B}$."
- Students could relate (or be asked to relate) Zoe's equation to familiar calculations involving whole numbers, such as 3 × 45 = 3 × 40 + 3 × 5, in which calculating a product involves adding terms.
- Similarly, students could think of, or be asked to think of, cases such as 7 × 99 = 700 - 7 in which calculating a product involves subtracting terms; or cases such as 88 ÷ 2, in which calculating a quotient implicitly or explicitly involves adding or subtracting terms.

## Related Math Milestones tasks

**4:12**

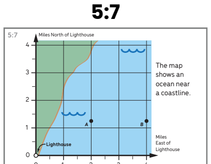
4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?


**5:14**

5:14 Brandon was reading his math book. He saw the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$. He said, "I don't get it—where did the 3 and the $\frac{3}{8}$ come from?" Write an explanation that could answer Brandon's question.

**5:7**

5:7




The map shows an ocean near a coastline.

Shipwrecks are at locations A $(2, 1\frac{1}{4})$ and B $(4, 1\frac{1}{4})$. Shipwrecks are also at locations C $(4, 3\frac{1}{2})$ and D $(2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle ABCD. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in $mi^2$?

**3:10**

3:10 Alice forgot what 7 × 8 equals. Alice knows that 5 × 8 = 40 and 2 × 8 = 16.

**(1)** Write a sentence to tell Alice how she can find the value of 7 × 8 by using the two facts she knows.
**(2)** Draw a diagram that could help Alice understand why your method works.
**(3)** Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Task **4:12 Super Hauler Truck** involves a multiplicative comparison leading to multiplying a fraction by a whole number in context. Because the fraction being multiplied is a mixed number, there are opportunities to apply the distributive property.

In later grades, task **5:14 Brandon's Equation** extends the reasoning from part (2) of task 4:5 to an equation in which the multiplier of $4\frac{1}{2}$ is a fraction $(\frac{3}{4})$ rather than a whole number (6). Task **5:7 Shipwrecks** involves calculating a product of mixed numbers in context.

### Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:5? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:5? In what specific ways do they differ from 4:5?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:5 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

In earlier grades, task **3:10 Alice's Multiplication Fact** involves applying the distributive property as part of the process of learning the multiplication table. This task also involves viewing expressions as objects with structure that can be interpreted—not just as instructions to calculate a resulting value.

## Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

### Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

### Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

### Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?