4:1 A Tablespoon of Oil

Teacher Notes

Math Milestones™

Central math concepts

Based on interviews with middle-grades students,[†] education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

The only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations, so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

For multiplication and division specifically, the mathematical relationship between the operations is that $C \div A$ is the unknown factor in $A \times ? = C$. Therefore, problems involving division also implicitly involve multiplication, because division finds an unknown factor. This is why a division calculation is checked by multiplying.

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between multiplication and division can play out in situations in conceptually distinct ways, including:[‡]

4:1

A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ______________

Answer: ______________

Answer

$3 \times m = 15$, $15 \div 3 = m$, or another equivalent equation. A teaspoon holds 5 ml of olive oil.

Click here for a student-facing version of the task.

Refer to the Standards

4.OA.A.2; MP.1, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

The numbers 15 and 3 in the task are intended to make the fluency demands of the task low so that attention can focus on the relationships in the situation.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:1? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:1? In what specific ways do they differ from 4:1?

- **Equal Groups:** Product Unknown, Group Size Unknown, and Number of Groups Unknown
- **Arrays:** Product Unknown, Number of Rows Unknown, and Number of Columns Unknown
- **Compare:** Size of Larger Quantity Unknown, Size of Smaller Quantity Unknown, and Multiplier Unknown

In particular, the situation type in task 4:1 is called "Compare with Smaller Unknown." It is a Compare situation because multiplication is being used to compare the capacity of a tablespoon with the capacity of a teaspoon. And the situation is "Compare with Smaller Unknown" because the initially unknown quantity is the capacity of a teaspoon (which is less than the capacity of a tablespoon).

Multiplicative comparison represents an advance on grade 3 multiplicative thinking. In multiplication and division word problems in grade 3, students multiplied whole numbers to find the total number of objects when the objects were grouped equally or were arrayed in rows and columns (including cases where the objects in rows and columns were square units), and students divided whole numbers to find an unknown factor in such situations (unknown group size, unknown number of groups, or unknown length measure). In grade 4, students will apply and extend their thinking about multiplication and division to solve problems of multiplicative comparison. These problems involve the idea of times-as-many/times-as-much, which is the idea that the product $A \times B$ refers to a quantity that is A times as many/times as much as the quantity B.

As a conceptual step beyond the idea of equal-groups, the idea of times-as-many/times-as-much may be an important step closer to the grade 5 idea of multiplication as a scaling operation that magnifies or shrinks a quantity. The conceptual or perhaps linguistic difference between times-as-many and times-as-much is that times-as-*many* applies most directly to discrete objects, like bowling balls or boats. Times-as-*much* applies most directly to substances that can be measured out and repeatedly subdivided, like a quantity of fluid or an interval of time. Often, the measures of these substances are fractions. Depending on the context, times-as-much language could involve phrases like "as long as," "as far as," "as tall (or short) as," "as heavy as," "as expensive as," and so on.

Task 4:1 asks not only for the final answer but also for an equation model. An equation model is requested because compared to the answer alone ("5 ml"), an equation model is better evidence that students have comprehended the situation and its quantitative relationships. The equation model records the situation's mathematical structure so that students can discuss and reflect on it. Equivalent forms of the equation can also show the relationship between multiplication and division.

It may happen that not all students write the same equation. That is an opportunity to discuss correspondences between equations and relationships to the situation described in the problem. The *Progression* document stresses that "[i]f textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:1 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 3, p. 23 of *Progressions for the Common Core State Standards in Mathematics (draft)*: *K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

representation to a context, and not the representation separated from its context" (p. 13).

Some equation models describe a situation in an algebraic way, such as $3 \times m = 15$, (these are called *situation equations*); students may then rewrite a situation equation in a form more conducive to determining the unknown number, such as $15 \div 3 = m$ (this is called a *solution equation*). As observed in the *Progression* document,

> Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. (p. 13)

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: recall of single-digit products; writing situation equations and solution equations; and using a tape diagram or other representation.

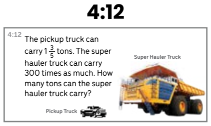
Extending the task

How might students drive the conversation further?

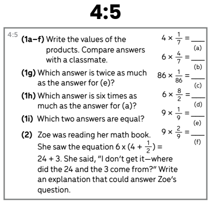
- Students who wrote different equation models could compare them and identify correspondences between the equations and each other, and the equations and the quantities and relationship in the situation.
- Students could discuss the claim that a teaspoon holds $\frac{1}{3}$ of what a tablespoon holds, basing their discussion on number lines or tape diagrams.

Related Math Milestones tasks

4:12


4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

4:5


4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} =$ ___ (a)

$6 \times \frac{4}{7} =$ ___ (b)

$86 \times \frac{1}{86} =$ ___ (c)

$6 \times \frac{8}{2} =$ ___ (d)

$9 \times \frac{1}{9} =$ ___ (e)

$9 \times \frac{2}{9} =$ ___ (f)

(1g) Which answer is twice as much as the answer for (e)?

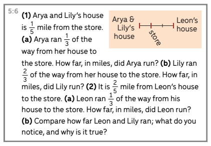
(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

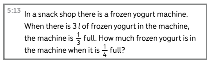
Task **4:12 Super Hauler Truck** is a word problem involving multiplicative comparison (the situation type is Compare with Larger Unknown; see Table 3, p. 23 of the *Progression* document). Task **4:5 Fraction Products and Properties** concentrates on the concepts of multiplying a fraction by a whole number, a step along the way to multiplying fractions in grade 5 and the idea of multiplication as scaling.

5:6



5:6 **(1)** Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?

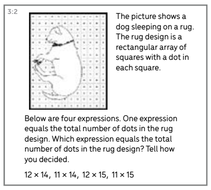
5:13



5:13 In a snack shop there is a frozen yogurt machine. When there is 3 l of frozen yogurt in the machine, the machine is $\frac{1}{3}$ full. How much frozen yogurt is in the machine when it is $\frac{1}{4}$ full?

In later grades, task **5:6 Corner Store** involves products of unit fractions and products of other fractions in context, and **task 5:13 Frozen Yogurt Machine** involves both multiplying and dividing with fractions.

3:2

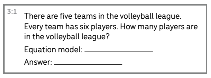


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

$12 \times 14,\ 11 \times 14,\ 12 \times 15,\ 11 \times 15$

3:1

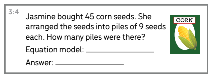


3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?

Equation model: ____________

Answer: ____________

3:4

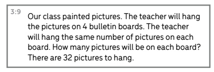


3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?

Equation model: ____________

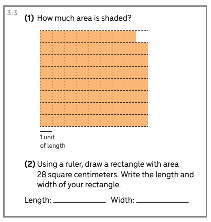
Answer: ____________

3:9



3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

3:3



3:3 **(1)** How much area is shaded?

(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ____________ Width: ____________

In earlier grades, task **3:2 Hidden Rug Design** centers on the equal-groups concept of multiplication in an array context, while tasks **3:1 Volleyball Players**, **3:4 Corn Seeds**, and **3:9 Bulletin Board Pictures** are word problems centered on equal-groups concepts of multiplication and/or division. Multiplication is useful in task **3:3 Length and Area Quantities**.

4:1 A Tablespoon of Oil
Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:2 Multi-Digit Division Concepts

Teacher Notes

Central math concepts

Asked to calculate the quotient 959 ÷ 7, a mischievous person might immediately answer, “Nine hundred fifty-nine sevenths.” That answer is correct! Unfortunately, it doesn’t tell us everything about the quotient we might need to know, such as what its tens digit is. Long division and its variations are useful algorithms for expressing the quotient of two multi-digit numbers as a multi-digit number. These algorithms also present opportunities for practicing and deepening student understanding of earlier mathematics such as place value to hundreds.

In part (1) of task 4:2, the dividend 959 is represented as the measure of a rectangular area; the divisor 7 is represented as a length measure for one dimension of the rectangle. Then the quotient, 137, corresponds to the length measure of the other dimension of the rectangle. Area ÷ *length* = *length*. If we view the rectangle with an unknown total length as a missing factor problem, 7 × ? = 959, then the unknown factor is the quotient 959 ÷ 7. Finding the base-ten digits of that quotient is what the division problem in part (3) is doing.

The distributive property $a \times (b + c) = a \times b + a \times c$ is fundamental not only to multiplying multi-digit numbers, but also to dividing them. In task 4:2, the division algorithm has parceled 959 into a sum of three terms, 959 = 700 + 210 + 49. The three terms have a common factor of 7:

$$959 = 7 \times 100 + 7 \times 30 + 7 \times 7.$$

Applying the distributive property to the right-hand side, we obtain

$$959 = 7 \times (100 + 30 + 7)$$

from which we can read the digits of the quotient, 137. We could express the multiplicative relationships above using division, too:

$$959 \div 7 = (700 + 210 + 49) \div 7$$

$$959 \div 7 = 700 \div 7 + 210 \div 7 + 49 \div 7$$

and this is a sense in which division could be said to distribute over addition.

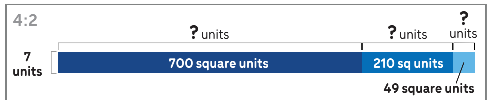
An interesting feature of the standard long division algorithm is that it returns the base-ten digits in succession at each successive step. However, from the standpoint of the distributive property, nothing requires that the decomposition of 959 be in base-ten units. The distributive property is true independently of the base-ten system for writing numbers. So for example, we might have

$$959 \div 7 = (350 + 420 + 189) \div 7$$

$$\begin{aligned} 959 \div 7 &= 350 \div 7 + 420 \div 7 + 189 \div 7 \\ &= 50 + 60 + 27 \\ &= 137. \end{aligned}$$

The partial quotients algorithm makes use of this flexibility in the distributive property.

4:2



(1) Find the three missing lengths and write them on the diagram. Compare answers with a classmate.

(2) What is the total area of the diagram?

(3) Look for connections between the diagram and the division problem. What connections can you see?

$$\begin{array}{r} 137 \\ 7\overline{)959} \\ -700 \\ \hline 259 \\ -210 \\ \hline 49 \\ -49 \\ \hline 0 \end{array}$$

Answer

(1) From left to right, the missing lengths are 100 units, 30 units, and 7 units. **(2)** 959 square units. **(3)** Answers will vary. Possible connections between the diagram and the division problem could include that the height of the rectangle is 7 units, and the divisor is 7; that the area measures in square units (700, 210, and 49) appear in the algorithm steps as partial products; that those partial products are the same products that give the shaded areas (7 × 100, 7 × 30, and 7 × 7); that the quotient is the sum of the length measures 100, 30, and 7; that the total area of the rectangle is 700 + 210 + 49 = 959, which is the dividend; that the algorithm step 959 - 700 = 259 corresponds to subtracting the darkest-shaded area from the total area.

Click here for a student-facing version of the task.

Refer to the Standards

4.NBT.B.6; MP.2, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Eventually, the kinds of reasoning processes involved in task 4:2 will cease to be the focus of a student's work with multi-digit division, and calculating quotients of whole numbers will become a routine exercise in procedural fluency. That progression takes time, however, and it requires a healthy mix of fluency practice coordinated with reasoning and conversational sense-making.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value to hundreds; multiplying with multiples of ten; multiplying by a single-digit factor; subtracting with three-digit numbers; and reasoning based on area concepts and the distributive property.

Extending the task

How might students drive the conversation further?

- If students normally calculate a quotient like 959 ÷ 7 using a different algorithm than the one shown (for example, partial quotients), they could ask if there is a way to change the diagram so that it connects to their work. For example, the boundaries between the shaded regions could be moved, and/or additional boundaries could be created.
- Students could ask how the diagram might be changed to match a problem with a remainder; the problem 960 = 7 × 137 + 1 could be considered.
- Students could brainstorm the most common careless errors ("bugs") a student might make in long division. Then students could formulate a tip for avoiding or finding and correcting the error in other long division problems.

Related Math Milestones tasks

4:11

4:11 A cook in the school kitchen uses 6 oz of cheese to make a pizza. The kitchen has 45 lb of cheese. How many pizzas will that make?

4:6

4:6 Grandpa took a jar of pennies to the bank. He said, "I'd like nickels for this, please." The bank teller poured the pennies into a counting machine. "Eighty-seven dollars and forty-two cents," said the teller. **(1)** How many nickels did Grandpa get? **(2)** Check your answer with an estimate.

4:10

4:10 Write the values of the products and quotients. Check the quotients by multiplying.

Mentally: 40 × 20; 30 × 11; 12 × 60; 5 × 19; 480 ÷ 8

With pencil and paper: $6{,}132 \times 6$; 48×39; $7\overline{)8{,}722}$

Task **4:11 School Kitchen** could involve finding a quotient of a three-digit dividend (which is a multiple of ten) with a one-digit divisor. Task **4:6 Jar of Pennies** could involve finding a quotient or whole-number quotient of a four-digit dividend and a one-digit divisor. Task **4:10 Calculating Products and Quotients** includes a multi-digit division problem.

5:5

5:5 Write the requested values.

4087 × 53 = ?	$\frac{1}{10} \div 10 = ?$	0.4 × 0.9 = ?
246 × 914 = ?	$\frac{7}{8} \times \frac{5}{3} = ?$	0.75 ÷ 0.01 = ?
9744 ÷ 12 = ?	8 × ? = 73	0.63 ÷ 0.3 = ?
1461 ÷ 6 = ?	$3 \div \frac{1}{8} = ?$	0.86 + 0.4 = ?
4 − (8 − 4) = ?	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	0.72 − 0.17 = ?
	$\frac{1}{3} \div (6 \times 5) = ?$	0.02 + 0.2 = ?
		0.8 − 0.55 = ?
		637 − 1.31 = ?

5:10

5:10 **(1)** Solve: $\frac{1}{3} = 0.1 + ?$
(2) Is there a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$? If you think so, find such a number. If you think there is no such number, explain why.
(3) Show one of the above problems and its solution on a number line.

In later grades, task **5:5 Calculating** includes multi-digit division problems. Task **5:10 Number System, Number Line** (part (1)) involves a problem in which one operates with numbers that are initially given in different formats (one fraction, one decimal).

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

To prompt students to make connections, students might first be asked to circle a three-digit number that they can see in both the diagram and the algorithm.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:2? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:2? In what specific ways do they differ from 4:2?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:2 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

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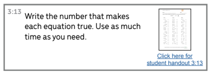
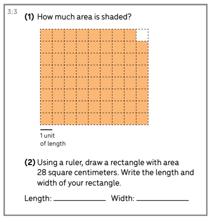
3:13



3:3



In earlier grades, task **3:13 Fluency within the Multiplication Table** involves mental calculation of quotients related to single-digit products. Task **3:3 Length and Area Quantities** involves concepts of area.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:3 Pencil Data

Teacher Notes

Central math concepts

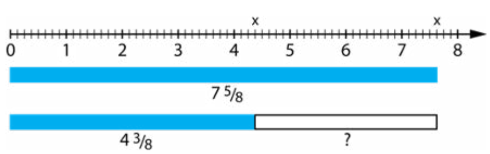
Students' data work in the upper-elementary grades concentrates on measurement data displayed in line plots.[†] In a line plot like the one shown in task 4:3, the "x" symbols are the individual data points.[‡]

The partial number line diagram in a line plot corresponds to the scale on the measurement tool that was used to generate the data. In task 4:3, the measurements are lengths in inches, but if the measurements had been liquid volumes, for example, then the units on the scale would be liters or another unit of liquid volume.

As for the vertical scale on a line plot, a vertical scale isn't shown on the line plot in task 4:3, but if a vertical scale had been shown, then it would be a count scale, meaning that the tick marks on the vertical scale would be the numbers 0, 1, 2, 3, and so on, indicating the number of observations above each tick mark.

Interpreting a line plot involves grasping the correspondence between an "x" symbol or dot, its horizontal position on the measurement scale, and what fact about the situation is being thereby recorded. For example, the leftmost "x" symbol records the fact that one of the pencils was $4\frac{3}{8}$ inches long.

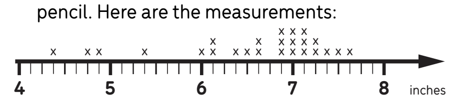
From the individual data points (the "x" symbols), students can use counting to determine the total number of observations, 25, which is one numerical summary of the data (part (1)).



Students' work in representing and analyzing measurement data connects directly to their growing number sense of fractions and to their increasing ability to use addition and subtraction with fractions to solve problems in context. In task 4:3, students use a partial number line diagram marked in eighths. They use subtraction to determine the answer to a Compare word problem (part (2); see figure). And they use addition and/or subtraction to determine the answer to a Put Together/Take Apart word problem with Both Addends Unknown. (See Table 2 in the relevant Progression document.[§]) There are close connections in every elementary grade between students' data work and their expanding use of numbers and operations in context; see Table 1, p. 4 of the relevant *Progression* document for a list of these connections in grades K–5.

As the *Guidelines for Assessment and Instruction in Statistics Education Report* notes, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning."[¶] Thus as the *Progression* document notes, "students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the context they represent" (p. 3).

4:3 Everyone in class measured the length of their pencil. Here are the measurements:



(1) How many pencils were measured?

(2) How much longer was the longest pencil than the shortest pencil?

(3) Could two of the pencils be laid end to end to make a total length of 1 foot?

Answer

(1) 25. **(2)** $3\frac{1}{4}$ inches (or $3\frac{2}{8}$ inches). (The longest pencil was $7\frac{5}{8}$ inches long, and the shortest pencil was $4\frac{3}{8}$ inches long; $7\frac{5}{8} - 4\frac{3}{8} = 3\frac{2}{8}$.) **(3)** Yes, because $4\frac{3}{8} + 7\frac{5}{8} = 11 + \frac{8}{8} = 12$.

Click here for a student-facing version of the task.

Refer to the Standards

4.MD.B.4; MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

- The situation types involved in the task are "Compare with Difference Unknown" and "Put Together/Take Apart with Both Addends Unknown." Students first encounter these situations with whole numbers in the primary grades, and they revisit them with fractional quantities in the upper-elementary grades.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: using number lines with fractions; calculating sums and differences of mixed numbers with equal denominators; and using addition and subtraction to solve problems in context.

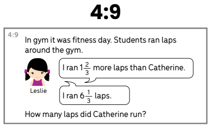
Extending the task

How might students drive the conversation further?

- Students could ask additional questions about the data, such as "How long would all the pencils be if they were all laid end to end?"
- Students could consider what the resulting line plot display of the data might look like in a week or two, imagining that everyone keeps their pencil but sharpens it from time to time.

Related Math Milestones tasks

4:9

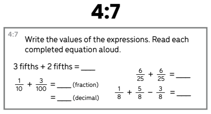

4:9 In gym it was fitness day. Students ran laps around the gym.

Leslie: I ran $1\frac{2}{3}$ more laps than Catherine.

I ran $6\frac{1}{3}$ laps.

How many laps did Catherine run?

4:7


4:7 Write the values of the expressions. Read each completed equation aloud.

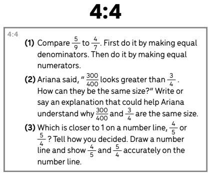
3 fifths + 2 fifths = ____

$\frac{1}{10} + \frac{3}{100} =$ ____ (fraction) = ____ (decimal)

$\frac{6}{25} + \frac{6}{25} =$ ____

$\frac{1}{8} + \frac{5}{8} - \frac{3}{8} =$ ____

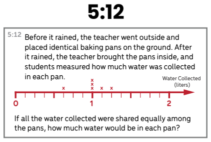
4:4


4:4 **(1)** Compare $\frac{5}{9}$ to $\frac{4}{7}$. First do it by making equal denominators. Then do it by making equal numerators.

(2) Ariana said, "$\frac{300}{400}$ looks greater than $\frac{3}{4}$. How can they be the same size?" Write or say an explanation that could help Ariana understand why $\frac{300}{400}$ and $\frac{3}{4}$ are the same size.

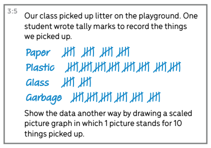
(3) Which is closer to 1 on a number line, $\frac{4}{5}$ or $\frac{5}{4}$? Tell how you decided. Draw a number line and show $\frac{4}{5}$ and $\frac{5}{4}$ accurately on the number line.

5:12


5:12 Before it rained, the teacher went outside and placed identical baking pans on the ground. After it rained, the teacher brought the pans inside, and students measured how much water was collected in each pan.

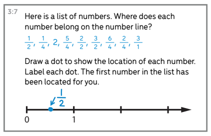
If all the water collected were shared equally among the pans, how much water would be in each pan?

3:5


3:5 Our class picked up litter on the playground. One student wrote tally marks to record the things we picked up.

Show the data another way by drawing a scaled picture graph in which 1 picture stands for 10 things picked up.

3:7


3:7 Here is a list of numbers. Where does each number belong on the number line?

$\frac{1}{2}, \frac{1}{4}, 2, \frac{5}{4}, \frac{2}{2}, \frac{3}{2}, \frac{6}{4}, \frac{2}{4}, \frac{3}{1}$

Draw a dot to show the location of each number. Label each dot. The first number in the list has been located for you.

Task **4:9 Fitness Day** involves a Compare situation with mixed-number quantities, as task 4:3 does. Task **4:7 Fraction Sums and Differences** involves fraction calculations with equal denominators, with a connection to unit thinking. Fractions on a number line are involved in task **4:4 Comparing Fractions with Equivalence** (part (3)).

In later grades, task **5:12 Rain Measurements** involves a line plot for measurement data and a calculation that prefigures measures of center and the study of distributions in grade 6.

In earlier grades, task **3:5 Playground Cleanup** focuses on representing a set of categorical data, which is the other major data type in the elementary grades. Task **3:7 Locating Numbers on a Number Line** involves whole numbers and fractions, including fractions equal to whole numbers.

Additional notes on the design of the task (continued)

- Part (1) is intended to orient students to the situation and to the connection between the line plot and the facts which it records.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:3? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:3? In what specific ways do they differ from 4:3?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:3 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2011, June 20). *Progressions for the Common Core State Standards in Mathematics (draft): K–3, Categorical Data; Grades 2–5, Measurement Data* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ In line plots generated by technology, data points are often marked by small filled circles, or "dots." Apart from that cosmetic feature, the terms *line plot* and *dot plot* are synonymous.

§ Common Core Standards Writing Team. (2011, May 29). *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

¶ The Guidelines for Assessment and Instruction in Statistics Education Report was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:4 Comparing Fractions with Equivalence

Teacher Notes

Central math concepts

The fraction equivalence principle $\frac{a}{b} = \frac{n \times a}{n \times b}$ is one of the most powerful reasoning tools and procedural tools in students' study of fractions. Far from being about "cancelling" at this grade, the fact that $\frac{a}{b}$ and $\frac{n \times a}{n \times b}$ have the same value (are the same size, occupy the same point on a number line) is a reasoned conclusion at this grade, based on conceptual understanding of the inverse relationship between the sizes of the parts $\frac{1}{b}$ and $\frac{1}{n \times b}$ and the number of those parts (a and $n \times a$)[†]. Part (2) of task 4:4 centers most directly on this multiplicative relationship, since it includes two equivalent fractions with a readily apparent multiplying factor in the numerator and denominator, $\frac{3}{4}$ and $\frac{300}{400}$. Meanwhile part (1) of the task leverages the power of the equivalence principle to compare two fractions in a case where using benchmark fractions would be difficult. Part (3) of the task uses the number line as the setting for integrating two essential fraction concepts: fraction size and fraction location on the number line. Students can also use fraction equivalence to help settle the question in this part of the task.

Students are often taught to remember procedures for comparing fractions, such as by "cross-multiplying" and comparing the resulting whole-number products. From the standpoint of mathematics as a discipline, procedures like that can be antithetical to the disciplinary values of mathematics if the rules aren't authentically accessible to justification and deliberation by students. From the standpoint of equity, rules and recipes delivered from "on high" devolve mathematical authority away from students and can thereby damage an equitable discourse community. From the standpoint of pragmatism, on-demand procedural fluency with exotic fraction comparisons just isn't important for elementary-school students. From the standpoint of achievement, rules that aren't understood can be harder to remember when needed. Finally, from the standpoint of mathematical content, reasoning about fractions ought to cast fractions in the starring role, whereas cross-multiplying concentrates attention on whole numbers. Observe that the answers to part (1) of task 4:4 are statements about fractions.

- The comparison $\frac{35}{63} < \frac{36}{63}$ involves the idea that 35 of the quantity $\frac{1}{63}$ is less than 36 of that same quantity.
- The comparison $\frac{20}{36} < \frac{20}{35}$ involves the idea that $\frac{1}{36}$ is less than $\frac{1}{35}$, and 20 of a smaller quantity is less than 20 of a larger quantity.

Whole numbers here are serving as multipliers of unit fractions, so that for example the phrase "35 of a quantity $\frac{1}{63}$" refers to a fractional quantity. This is different from saying $\frac{5}{9} < \frac{4}{7}$ "because $5 \times 7 < 9 \times 4$"...a statement

4:4

(1) Compare $\frac{5}{9}$ to $\frac{4}{7}$. First do it by making equal denominators. Then do it by making equal numerators.

(2) Ariana said, "$\frac{300}{400}$ looks greater than $\frac{3}{4}$. How can they be the same size?" Write or say an explanation that could help Ariana understand why $\frac{300}{400}$ and $\frac{3}{4}$ are the same size.

(3) Which is closer to 1 on a number line, $\frac{4}{5}$ or $\frac{5}{4}$? Tell how you decided. Draw a number line and show $\frac{4}{5}$ and $\frac{5}{4}$ accurately on the number line.

Answer

(1) *Equal denominators comparison*: $\frac{5}{9} < \frac{4}{7}$ because $\frac{35}{63} < \frac{36}{63}$. (This comparison involves the idea that 35 of the quantity $\frac{1}{63}$ is less than 36 of that same quantity.)

Equal numerators comparison: $\frac{5}{9} < \frac{4}{7}$ because $\frac{20}{36} < \frac{20}{35}$. (This comparison involves the idea that $\frac{1}{36}$ is less than $\frac{1}{35}$, and 20 of a smaller quantity is less than 20 of a larger quantity.)

(2) Answers will vary. One kind of explanation involves the idea that 3 of a quantity that is a hundred times greater equals 300 of a quantity that is a hundred times smaller. Answers may include such explanatory techniques as emphasizing the meaning of the numerator and denominator in $\frac{300}{400}$; creating a simple context for the numbers in Ariana's comparison (for example, $\frac{1}{4}$ of \$400 is \$100, so $\frac{3}{4}$ is \$300, and $\frac{1}{400}$ of \$400 is \$1, so $\frac{300}{400}$ is \$300); and writing an explanation based on a drawn number line that indicates (conceptually) the equivalence between $\frac{3}{4}$ and $\frac{300}{400}$.

about whole numbers that is unattached to the fraction quantities in the problem or their unit fractions.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: using unit fractions; recall of single-digit products; and basing reasoning on math diagrams.

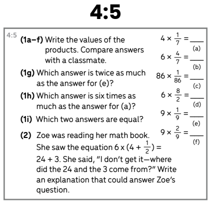
Extending the task

How might students drive the conversation further?

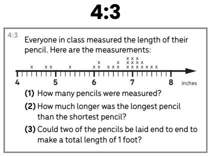
- Students could notice in part (1) that the numbers 35 and 36 "made all the difference" in comparing $\frac{5}{9}$ and $\frac{4}{7}$, using both the numerator approach and the denominator approach. Can this observation be generalized? When comparing $\frac{19}{25}$ and $\frac{16}{21}$ using the numerator approach and the denominator approach, what numbers "make all the difference"?
- Students could consider another version of Ariana's question in part (2), this time involving the fractions $\frac{377}{477}$ and $\frac{3}{4}$. Are these fractions equivalent? Why or why not?
- Building on the conclusions of part (3), students could consider additional cases that fit the pattern of $\frac{4}{5}$ and $\frac{5}{4}$, such $\frac{2}{3}$ and $\frac{3}{2}$, $\frac{7}{8}$ and $\frac{8}{7}$, etc. Can the conclusion of part (3) be generalized?

Related Math Milestones tasks

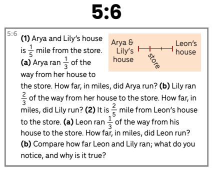
4:5



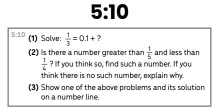
4:3



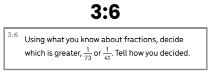
5:6



5:10



3:6



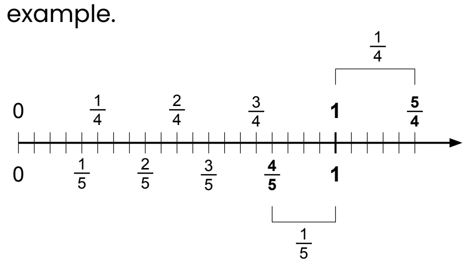
Fraction equivalence sheds light on some parts of task **4:5 Fraction Products and Properties** (such as the fact that 6 × $\frac{8}{2}$ = 6 × 4). A number line appears in task **4:3 Pencil Data** as a measurement scale, and number lines could be used to discuss the mathematics in several other tasks.

In later grades, fraction equivalence could be used to solve or illuminate the mathematics in several tasks, for example, **5:6 Corner Store**. Task **5:10 Number System, Number Line** involves a number line as part of synthesizing fraction and decimal operations and reasoning.

In earlier grades, task **3:6 Unit Fraction Ideas** is a conceptual task about this foundational fraction concept.

Answer (continued)

(3) Explanations may vary but could involve the idea that $\frac{1}{5}$ is smaller than $\frac{1}{4}$, so that 1 - $\frac{4}{5}$ is less far to the left of 1 than 1 + $\frac{1}{4}$ is to the right of 1. Other explanations could involve finding equivalent fractions, including equivalent fractions with denominator 100. Diagrams may also vary; see the example.



Click here for a student-facing version of the task.

Refer to the Standards

4.NF.A; MP.3, MP.6, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

Students create a wide variety of mathematical products in task 4:4 by generating equivalent fractions, making statements of comparison, writing or speaking mathematical reasoning and conclusions, and creating mathematical representations.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:4? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:4? In what specific ways do they differ from 4:4?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:4 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† This understanding in turn is based on perhaps the single most foundational fraction concept, that of the unit fraction (for more on this concept see the Teacher Notes for task **3:6 Unit Fraction Ideas**).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

Central math concepts

Parts (1a–f). Extending multiplication and division from whole numbers to fractions may be the most important content progression of grades 3-5. This progression traces a substantial evolution in students' concepts about numbers and operations. The need to support students in making that conceptual evolution raises important questions: What aspects of earlier thinking about multiplying whole numbers will remain helpful when making sense of a product involving fractions? What new ways of thinking will be helpful? And what mathematical representations introduced during whole-number multiplication work are best suited to supporting the transition from multiplying and dividing whole numbers to multiplying and dividing all numbers?

An early step in the progression of multiplication and division operations is making sense of multiplying a fraction by a whole number. The key ideas involved are listed in the table. An important aspect of working with these ideas is viewing expressions like $86 \times \frac{1}{86}$ not just as instructions to calculate a resulting value, but also as objects with structure that can be interpreted while delaying that evaluation.

Key Ideas Leading to Multiplying Fractions by Whole Numbers.

I. The unit fraction idea. (Example: $\frac{1}{5}$)

- $\frac{1}{5}$ mile means 1 part of a partition of 1 mile into 5 equal parts.
- 5 parts of size $\frac{1}{5}$ make 1 mile, that is, $5 \times \frac{1}{5} = 1$.
- Expressing the reasoning more generally leads to $B \times \frac{1}{B} = 1$.

II. Building general fractions from unit fractions. (Example: $2 \times \frac{1}{5}$)

- $\frac{2}{5}$ mile means 2 parts of size $\frac{1}{5}$ mile, or twice as much as $\frac{1}{5}$ mile.
- That is to say, $\frac{2}{5}$ means $2 \times \frac{1}{5}$.
- In terms of unit thinking, $\frac{2}{5}$ is 2 fifths, or 2 units of one-fifth.
- Expressing the reasoning more generally, $\frac{A}{B}$ means $A \times \frac{1}{B}$ and $A \times \frac{1}{B} = \frac{A}{B}$.

III. Multiplying general fractions by whole numbers. (Example: $4 \times \frac{3}{5}$)

- $\frac{3}{5}$ means 3 parts of size $\frac{1}{5}$.
- 4 times as much as 3 parts of size $\frac{1}{5}$ amounts to 4×3 parts of size $\frac{1}{5}$.
- 4×3 parts of size $\frac{1}{5}$ equal the fraction $\frac{4 \times 3}{5}$. (By the principles of Row II.)
- In terms of unit thinking, 4 times (3 fifths) = (4 times 3) fifths. This is like the problem $4 \times 30 = 4 \times (3 \text{ tens}) = (4 \times 3)$ tens. The units of tens are analogous to the units of fifths. And the fundamental principle at work is the associative property of multiplication, $r \times (s \times t) = (r \times s) \times t$.
- Expressing the reasoning more generally, $A \times \frac{C}{D} = \frac{A \times C}{D}$.

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

(1g) Which answer is twice as much as the answer for (e)?

(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

$4 \times \frac{1}{7} =$ ___ (a)

$6 \times \frac{4}{7} =$ ___ (b)

$86 \times \frac{1}{86} =$ ___ (c)

$6 \times \frac{8}{2} =$ ___ (d)

$9 \times \frac{1}{9} =$ ___ (e)

$9 \times \frac{2}{9} =$ ___ (f)

Answer

(1) (a) $\frac{4}{7}$. (b) $\frac{24}{7}$ or $3\frac{3}{7}$. (c) 1. (d) $\frac{48}{2}, \frac{24}{1}$, or 24. (e) 1. (f) $\frac{18}{9}, \frac{6}{3}, \frac{2}{1}$, or 2. (g) $\frac{18}{9}, \frac{6}{3}, \frac{2}{1}$, or 2. (h) $\frac{24}{7}$ or $3\frac{3}{7}$. (i) 1 and 1 (from parts (c) and (e)). **(2)** Answers will vary. One kind of explanation involves calculating $6 \times 4\frac{1}{2} = 6 \times \frac{9}{2} = \frac{54}{2} = 27$, then recognizing that $27 = 24 + 3$.

A second kind of explanation involves an application of the distributive property: Because $6 \times 4 = 24$ and $6 \times \frac{1}{2} = 3$, we can multiply $6 \times (4 + \frac{1}{2})$ by adding the products: $6 \times (4 + \frac{1}{2}) = 6 \times 4 + 6 \times \frac{1}{2} = 24 + 3$. This reveals 24 and 3 as partial products. A third kind of explanation may take advantage of the bidirectionality of the equal sign by beginning the reasoning with 24 + 3 and working from right to left. (For example, 24 + 3 equals 3 times 8 + 1, and 8 + 1 is twice as much as $4 + \frac{1}{2}$, so altogether 24 + 3 equals 3×2 times $4 + \frac{1}{2}$.)

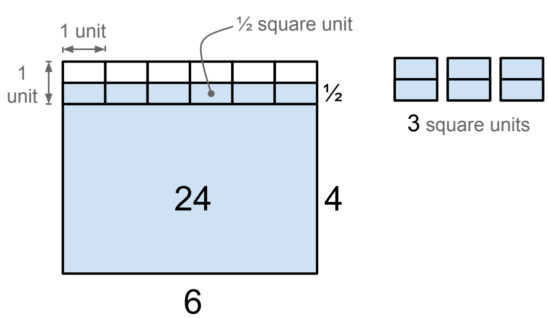
Part (2). One way in which we often use the distributive property, explicitly or implicitly, is to help us calculate a product of sums. For elementary-grades students, the first such problems arise when calculating products of multi-digit numbers; this is because every multi-digit number is a sum of terms. For example, 19 is a sum 10 + 9, which means we could calculate 5 × 19 mentally by thinking "It's 50 + 45," or "It's 100 - 5."

Upper-elementary students next apply the distributive property to products and sums that involve fractions. That development or progress is important because in middle grades, students will apply the distributive property to products and sums that involve not only whole numbers and fractions, but also rational numbers, real numbers, variables, and variable expressions. High school students will apply the distributive property not only to real numbers and variable expressions, but also to complex numbers, and perhaps to matrices. If the inner structure of arithmetic and algebra could be likened to a skeleton, then the distributive property would be the backbone.

In the symbol 378, there's nothing visible that tells us it refers to a sum; primary-grades students must learn to unpack 378 as 300 + 70 + 8. Similarly, upper-elementary students learn to unpack a symbol like 7.3 as $7 + \frac{3}{10}$. Mixed numbers are also sums, although this is not indicated by a symbol like $4\frac{1}{2}$. Fortunately, the symbol $4\frac{1}{2}$ is read aloud as "Four *and* one-half," where the word *and* supports correct interpretation of the symbol. Still, even high school students can sometimes accidentally misinterpret a symbol like $4\frac{1}{2}$ or have difficulty entering the number into a calculator. Their fluency in the conventions of algebra (in particular, the use of juxtaposition to indicate multiplication) can occasionally mislead them into thinking that $4\frac{1}{2}$ refers to the product $4 \times \frac{1}{2}$ rather than the sum $4 + \frac{1}{2}$.

A student who efficiently calculates $6 \times 4\frac{1}{2} = 6 \times \frac{9}{2} = \frac{54}{2} = 27$ has demonstrated a valuable fluency. Efficiently evaluating fraction products is important. Also important, especially as preparation for algebra, is being able to study expressions like the ones on either side of the equal sign in Zoe's equation, and *delay* evaluation of them in order to analyze the expression as an object with structure.

Thus, the most insightful explanation of the equation $6 \times 4\frac{1}{2} = 24 + 3$ arguably isn't simply to observe that the numerical values on either side of the equal sign are equal. For example, the same area models that illustrate the distributive property for whole numbers could be used to make sense of the numbers in Zoe's equation as partial products:



Answer (continued)

Answers may include such explanatory techniques as, for example: showing a math diagram, such as an area model; creating a simple word problem that makes sense of the numbers in Zoe's equation; and/or writing expressions and equations.

[Click here](#) for a student-facing version of the task.

Refer to the Standards

4.NF.B.4a, 4b; MP.3, MP.5, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- Questions (1a–f) can be answered solely using procedural knowledge. The intent of the follow-up questions (1g–i) is to invite students to notice patterns in the results and make sense of the products, for example by relating the size of the product to the size of the factors.
- Zoe's question specifically asks, "Where did the 24 and the 3 come from?"—not just "I don't get it," or "Is this equation true?" The phrasing is intended to invite sense-making about the equation. Implicit in Zoe's question, perhaps, is that the confusing numbers 24 and 3 are addends; so part of Zoe's confusion might be that a calculation like $6 \times 4\frac{1}{2}$, in which we were supposed to multiply, got turned into an addition problem.

To be sure, it is mathematically valid to explain Zoe's equation by observing that the numerical values on either side of the equal sign are equal—it's an ironclad proof. That explanation also demonstrates an understanding of the meaning of the equal sign. Thus, the purpose of task 4:5 isn't to differentiate between the students who explain things one way and the students who explain things the other way; the purpose is to bring different explanations together, and relate those explanations to one another, so that all students deepen their understanding and build a foundation for future learning.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: recall of single-digit products; multiplying $\frac{1}{2}$ by an even number; and basing multiplicative reasoning and distributive property reasoning on math diagrams.

Extending the task

How might students drive the conversation further?

- Building on the particular cases $9 \times \frac{1}{9} = 1$ and $86 \times \frac{1}{86} = 1$, students could propose, and create a justification for, a statement that generalizes from those cases, either verbally or by creating a mathematical statement such as "$B \times \frac{1}{B} = 1$ for any unit fraction $\frac{1}{B}$."
- Students could relate (or be asked to relate) Zoe's equation to familiar calculations involving whole numbers, such as $3 \times 45 = 3 \times 40 + 3 \times 5$, in which calculating a product involves adding terms.
- Similarly, students could think of, or be asked to think of, cases such as $7 \times 99 = 700 - 7$ in which calculating a product involves subtracting terms; or cases such as $88 \div 2$, in which calculating a quotient implicitly or explicitly involves adding or subtracting terms.

Related Math Milestones tasks

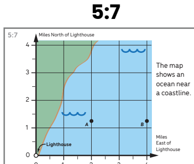
4:12

4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

5:14

5:14 Brandon was reading his math book. He saw the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$. He said, "I don't get it—where did the 3 and the $\frac{3}{8}$ come from?" Write an explanation that could answer Brandon's question.

5:7



5:7 The map shows an ocean near a coastline.

Shipwrecks are at locations A $(2, 1\frac{1}{4})$ and B $(4, 1\frac{1}{4})$. Shipwrecks are also at locations C $(4, 3\frac{1}{2})$ and D $(2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

3:10

3:10 Alice forgot what 7×8 equals. Alice knows that $5 \times 8 = 40$ and $2 \times 8 = 16$.

(1) Write a sentence to tell Alice how she can find the value of 7×8 by using the two facts she knows.
(2) Draw a diagram that could help Alice understand why your method works.
(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Task **4:12 Super Hauler Truck** involves a multiplicative comparison leading to multiplying a fraction by a whole number in context. Because the fraction being multiplied is a mixed number, there are opportunities to apply the distributive property.

In later grades, task **5:14 Brandon's Equation** extends the reasoning from part (2) of task 4:5 to an equation in which the multiplier of $4\frac{1}{2}$ is a fraction ($\frac{3}{4}$) rather than a whole number (6). Task **5:7 Shipwrecks** involves calculating a product of mixed numbers in context.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:5? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:5? In what specific ways do they differ from 4:5?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:5 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

In earlier grades, task **3:10 Alice's Multiplication Fact** involves applying the distributive property as part of the process of learning the multiplication table. This task also involves viewing expressions as objects with structure that can be interpreted—not just as instructions to calculate a resulting value.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:6 Jar of Pennies

Teacher Notes

Central math concepts

Whereas the question 8,742 ÷ 5 = ? poses a procedural task, the question "How many nickels did Grandpa get?" poses a modeling task. That's because significant interpretation is required to relate the mathematics chosen by the student to the situation at hand. An equation model for Grandpa's situation might read as follows:

$$8{,}742 = 5 \times 1{,}748 + 2 \qquad (*)$$

Each number in this equation has a meaning in the problem context: 8,742 is the value of Grandpa's pennies, in cents; 5 is the value of a nickel, in cents; 1,748 is the number of nickels Grandpa will get; and 2 is the number of pennies due back to Grandpa from the teller. Procedural skill is involved in determining the numbers 1,748 and 2, given the numbers 8,742 and 5; modeling skill is involved in interpreting those numbers in context.

The number 1,748 in Equation (*) is often called the "quotient" of 8,742 and 5, but in more precise language 1,748 would be called the *whole-number quotient* of 8,742 and 5. The quotient of 8,742 and 5 is not 1,748; it's $1{,}748\frac{2}{5}$. To see this, note that

$$\begin{aligned} 5 \times \left(1{,}748\tfrac{2}{5}\right) &= 5 \times \left(1{,}748 + \tfrac{2}{5}\right) \\ &= 5 \times 1{,}748 + 5 \times \tfrac{2}{5} \quad \text{(using the distributive property)} \\ &= 8{,}740 + 2 \\ &= 8{,}742 \end{aligned}$$

In other words, $1{,}748\frac{2}{5}$ is the number which, when multiplied by 5, yields 8,742. This shows that $8{,}742 \div 5 = 1{,}748\frac{2}{5}$, because $C \div A$ is the unknown factor in $A \times ? = C$. This relationship is a trustworthy guide to mathematical reasoning, used frequently beginning in grade 3 and continuing at least until grade 7. The relationship recurs in high school as well, with the study of rational expressions, rational functions, and complex numbers. Consistent with so fundamental and useful a principle, a student should always be able to check a quotient by multiplying.

Another possible equation model for Grandpa's situation might be:

$$(8{,}742 - 2) \div 5 = 1{,}748 \qquad (**)$$

This equation tells its own story about the situation, a story in which we could imagine Grandpa removing two pennies from the jar before handing it over to the bank teller. Then the value of the remaining pennies would equal the value of a whole number of nickels. (Maybe Grandpa made a lucky guess here!)

We can imagine partitioning a cup of flour into smaller and smaller quantities of flour, but we don't think that way about discrete quantities like lunch trays, marbles, subway seats, and school crossing guards. So in a word problem, the context determines the best answer to give—whether this be the exact quotient of two given numbers, the whole number quotient, the quotient rounded to the next greatest whole number, or even the remainder.

4:6 Grandpa took a jar of pennies to the bank. He said, "I'd like nickels for this, please." The bank teller poured the pennies into a counting machine. "Eighty-seven dollars and forty-two cents," said the teller. **(1)** How many nickels did Grandpa get? **(2)** Check your answer with an estimate.

Answer

(1) 1,748. **(2)** Estimates will vary. For example, an estimate of 2,000 could come from rounding \$87.42 to \$100, then reasoning that since \$1 is worth 20 nickels, \$100 is worth 2,000 nickels.

Click here for a student-facing version of the task.

Refer to the Standards

4.OA.A.3; MP.1, MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application, Procedural skill and fluency

Additional notes on the design of the task

- In the text of task 4:6, words that suggest division are absent. This is intended is to disincentivize "key word hunting" and also make room for equation models like the one in Equation (*) that involve multiplication and addition.
- Because of the lack of key words, it might be useful for students to act out the problem, one playing the part of Grandpa and another playing the part of the bank teller.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: mental calculation; finding quotients of multi-digit numbers; rewriting decimal quantities as whole number multiples of a smaller unit; and rounding and estimation.

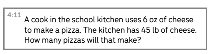
Extending the task

How might students drive the conversation further?

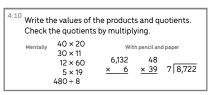
- Students could recommend that Grandpa look in his pockets for three more pennies.
- Students could judge that 1,748 nickels is an inconveniently large number of nickels and ask what Grandpa would have received instead if he had asked for quarters.
- Students might know that banks sometimes handle rolls of coins. A roll of nickels has a value of $2. How many nickels is that? If Grandpa had asked for the nickels in rolls, how many rolls would that be?

Related Math Milestones tasks

4:11


4:11 A cook in the school kitchen uses 6 oz of cheese to make a pizza. The kitchen has 45 lb of cheese. How many pizzas will that make?

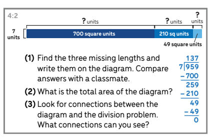
4:10


4:10 Write the values of the products and quotients. Check the quotients by multiplying.

Mentally: 40×20, 30×11, 12×60, 5×19, $480 \div 8$

With pencil and paper: $6{,}132 \times 6$, 48×39, $7\overline{)8{,}722}$

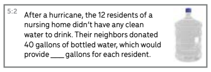
4:2



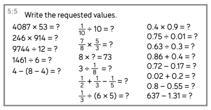
4:2 (1) Find the three missing lengths and write them on the diagram. Compare answers with a classmate.
(2) What is the total area of the diagram?
(3) Look for connections between the diagram and the division problem. What connections can you see?

Task **4:11 School Kitchen** involves division in context, with the added element that the given quantities have different units. Task **4:10 Calculating Products and Quotients** involves multi-digit procedures like those which could be used for task 4:6. Task **4:2 Multi-Digit Division Concepts** involves the standard multi-digit division algorithm with a conceptual focus.

5:2


5:2 After a hurricane, the 12 residents of a nursing home didn't have any clean water to drink. Their neighbors donated 40 gallons of bottled water, which would provide ____ gallons for each resident.

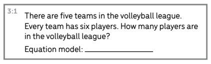
5:5


5:5 Write the requested values.

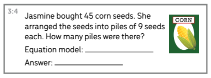
$4087 \times 53 = ?$	$\frac{1}{10} \div 10 = ?$	$0.4 \times 0.9 = ?$
$246 \times 914 = ?$	$\frac{7}{8} \times \frac{5}{3} = ?$	$0.75 \div 0.01 = ?$
$9744 \div 12 = ?$	$8 \times ? = 73$	$0.63 \div 0.3 = ?$
$1461 \div 6 = ?$	$3 \div \frac{1}{8} = ?$	$0.86 + 0.4 = ?$
$4 - (8 - 4) = ?$	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	$0.72 - 0.17 = ?$
	$\frac{1}{3} \div (6 \times 5) = ?$	$0.02 + 0.2 = ?$
		$0.8 - 0.55 = ?$
		$637 - 1.31 = ?$

In later grades, the progression of dividing with whole numbers comes to a close as the fraction $\frac{A}{B}$ becomes understood as the quotient $A \div B$; task **5:2 Water Relief** centers on this concept in context, and task **5:5 Calculating** includes problems involving division of whole numbers leading to a non-whole-number quotient.

3:1

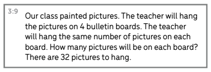

3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?
Equation model: ____________

3:4



3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?
Equation model: ____________
Answer: ____________

3:9


3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

In earlier grades, tasks **3:1 Volleyball Players**, **3:4 Corn Seeds**, and **3:9 Bulletin Board Pictures** form a kind of survey of the essential early meanings of multiplication and division; the requested equation models can support learning about the relationship between the operations.

Additional notes on the design of the task (continued)

- The artwork consisting of a collection of 5 pennies could be useful for prompting mathematical thinking about how to approach the problem. Students who don't know that nickels are worth 5 cents can be given this information.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:6? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:6? In what specific ways do they differ from 4:6?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:6 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

4:6 Jar of Pennies

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

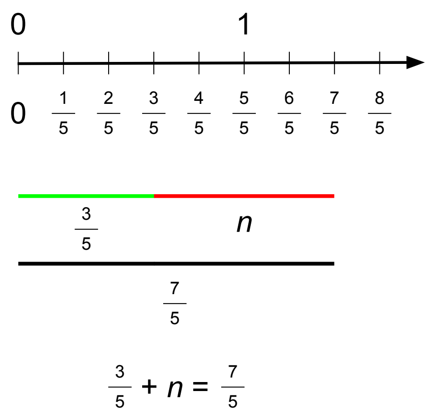
- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:7 Fraction Sums and Differences

Teacher Notes

Central math concepts

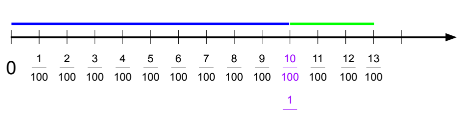
Extending addition and subtraction from whole numbers to fractions enables students to apply mathematics not only in relation to discrete objects, like bowling balls or boats, but also in relation to substances that can be measured out and repeatedly subdivided, like a quantity of fluid or an interval of time. Often, the measures of these substances are fractions. As students extend addition and subtraction from whole numbers to fractions, the meanings of the operations remain the same: adding remains an operation of joining or adding-to, and subtraction remains an operation of separating or taking-from. Also unchanged, then, is the relationship between addition and subtraction: just as with whole numbers, $C - A$ is the unknown addend in $A + \square = C$. Because a number line shows the continuum of all numbers, a number line can efficiently represent sums and differences of fractions (see figure, which shows the unknown addend problem $\frac{3}{5} + n = \frac{7}{5}$).



Unit fractions $\frac{1}{b}$ are the building blocks of fractions,† in the multiplicative sense that a general fraction $\frac{a}{b}$ is a copies of $\frac{1}{b}$ (written in symbols, $\frac{a}{b} = a \times \frac{1}{b}$), and also in the additive sense that in a problem like $\frac{3}{5} + \frac{2}{5} = ?$, the unit fraction $\frac{1}{5}$ can be thought of as a unit (one "fifth"), so that the sum $\frac{2}{5} + \frac{3}{5}$ can be thought of as 2 of the unit plus 3 of the unit, yielding 5 of the unit (5 fifths or $\frac{5}{5}$). Thanks to unit fractions, the unknown-addend problem shown in the figure, $\frac{3}{5} + n = \frac{7}{5}$, could be solved by thinking, "3 fifths plus how many fifths equals 7 fifths?" in somewhat the same way that students in first grade might have thought about the problem "3 eggs plus how many eggs is 7 eggs?"

Conceptual understanding of fraction addition and subtraction with equal denominators will support students in adding and subtracting fractions with unequal denominators in grade 5, because the core strategy for adding and subtracting with unequal denominators is to use the principle of fraction equivalence to replace the given problem with a problem in which the denominators are equal.

This strategy also applies to one of the sums in task 4:7, namely $\frac{1}{10} + \frac{3}{100}$. Using the principle of fraction equivalence $\frac{a}{b} = \frac{n \times a}{n \times b}$, the first addend $\frac{1}{10}$ is equal to $\frac{10 \times 1}{10 \times 10} = \frac{10}{100}$. So the sum $\frac{1}{10} + \frac{3}{100}$ equals $\frac{10}{100} + \frac{3}{100}$ (see the figure).



4:7 Write the values of the expressions. Read each completed equation aloud.

3 fifths + 2 fifths = ___

$\frac{1}{10} + \frac{3}{100}$ = ___ (fraction)

= ___ (decimal)

$\frac{6}{25} + \frac{6}{25}$ = ___

$\frac{1}{8} + \frac{5}{8} - \frac{3}{8}$ = ___

Answer

5 fifths or $\frac{5}{5}$ or 1; $\frac{13}{100}$, 0.13; $\frac{12}{25}$; $\frac{3}{8}$.

Click here for a student-facing version of the task.

Refer to the Standards

4.NF.B.3a–c, 4.NF.C.5, 6; MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Procedural skill and fluency

Additional notes on the design of the task

- Task 4:7 is designed to target conceptual understanding, even though it only asks for brief answers rather than asking for extended writing or making other language demands. Teachers can also question students about the thinking that led to their answers, individually or in a group setting (and students can question each other).

In terms of unit thinking, the problem $\frac{1}{10} + \frac{3}{100} = ?$ is like asking for the total value of 1 dime and 3 pennies, a problem we could solve by using a common unit of pennies for both values.

Task 4:7 also asks for the total $\frac{13}{100}$ in decimal notation as 0.13. Using decimal notation for fractions that are multiples of $\frac{1}{10}$ or multiples of $\frac{1}{100}$ is the opening for the full study of the place value system in grade 5.

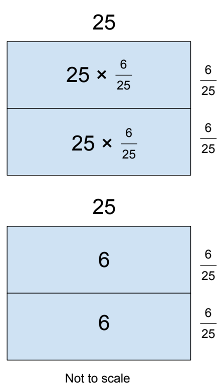
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: working with unit fractions; single-digit sums and differences; and understanding the meanings of the operations of addition and subtraction.

Extending the task

How might students drive the conversation further?

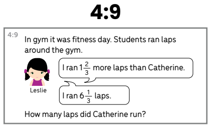
- Students could analyze an area model to understand that $25 \times (\frac{6}{25} + \frac{6}{25}) = 25 \times \frac{6}{25} + 25 \times \frac{6}{25}$. And since each addend is 6, the sum is 12; in other words, the result of $25 \times (\frac{6}{25} + \frac{6}{25})$ should be 12. Students can then check: is 12 the result you get from multiplying their answer to $\frac{6}{25} + \frac{6}{25} = ?$ by 12?
- Students could produce the area model and apply the reasoning to another case that they produce, like $7 \times (\frac{2}{7} + \frac{2}{7}) = 7 \times \frac{2}{7} + 7 \times \frac{2}{7} = 2 + 2 = 4$.



Not to scale

Related Math Milestones tasks

4:9

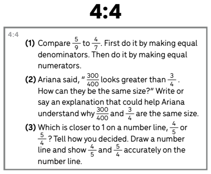

4:9 In gym it was fitness day. Students ran laps around the gym.

Leslie: I ran $1\frac{2}{3}$ more laps than Catherine.

I ran $6\frac{1}{3}$ laps.

How many laps did Catherine run?

4:4

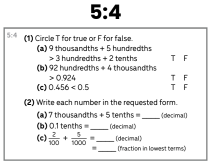

4:4 **(1)** Compare $\frac{5}{9}$ to $\frac{4}{7}$. First do it by making equal denominators. Then do it by making equal numerators.

(2) Ariana said, "$\frac{300}{400}$ looks greater than $\frac{3}{4}$. How can they be the same size?" Write or say an explanation that could help Ariana understand why $\frac{300}{400}$ and $\frac{3}{4}$ are the same size.

(3) Which is closer to 1 on a number line, $\frac{4}{5}$ or $\frac{5}{4}$? Tell how you decided. Draw a number line and show $\frac{4}{5}$ and $\frac{5}{4}$ accurately on the number line.

Task **4:9 Fitness Day** is a word problem involving addition and subtraction with fractions. Task **4:4 Comparing Fractions with Equivalence** involves the principle of fraction equivalence in connection with fraction size.

5:4


5:4 **(1)** Circle T for true or F for false.

(a) 9 thousandths + 5 hundredths > 3 hundredths + 2 tenths T F

(b) 92 hundredths + 4 thousandths > 0.924 T F

(c) 0.456 < 0.5 T F

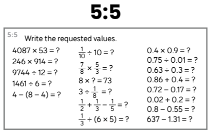
(2) Write each number in the requested form.

(a) 7 thousandths + 5 tenths = _____ (decimal)

(b) 0.1 tenths = _____ (decimal)

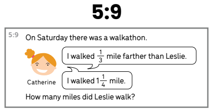
(c) $\frac{2}{100} + \frac{5}{1000}$ = _____ (decimal) = _____ (fraction in lowest terms)

5:5


5:5 Write the requested values.

$4087 \times 53 = ?$	$\frac{1}{10} \div 10 = ?$	$0.4 \times 0.9 = ?$
$246 \times 914 = ?$	$\frac{7}{8} \times \frac{5}{3} = ?$	$0.75 \div 0.01 = ?$
$9744 \div 12 = ?$	$8 \times ? = 73$	$0.63 \div 0.3 = ?$
$1461 \div 6 = ?$	$3 \div \frac{1}{8} = ?$	$0.86 + 0.4 = ?$
$4 - (8 - 4) = ?$	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	$0.72 - 0.17 = ?$
	$\frac{1}{3} \div (6 \times 5) = ?$	$0.02 + 0.2 = ?$
		$0.8 - 0.55 = ?$
		$637 - 1.31 = ?$

5:9

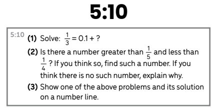

5:9 On Saturday there was a walkathon.

Catherine: I walked $\frac{1}{3}$ mile farther than Leslie.

I walked $1\frac{1}{4}$ mile.

How many miles did Leslie walk?

5:10


5:10 **(1)** Solve: $\frac{1}{3} = 0.1 + ?$

(2) Is there a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$? If you think so, find such a number. If you think there is no such number, explain why.

(3) Show one of the above problems and its solution on a number line.

In later grades, task **5:4 Place Value to Thousandths** situates problems like $\frac{2}{10} + \frac{5}{1000}$ in the context of decimal place value. Task **5:5 Calculating**

Additional notes on the design of the task (continued)

- "Read each completed equation aloud" is intended to leverage the way spoken names of fractions evoke units, as in "Six twenty-fifths plus six twenty-fifths equals twelve twenty-fifths." And in this case, the use of two sixes is intended to evoke a potentially familiar "doubles fact" from primary-grades addition work.

Curriculum connection

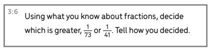
1. In which unit of your curriculum would you expect to find tasks like 4:7? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:7? In what specific ways do they differ from 4:7?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:7 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† See Zimba (2013), "Units, a Unifying Idea in Measurement, Fractions, and Base Ten."

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

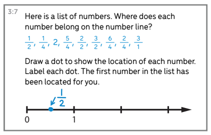
involves sums and differences of fractions with unequal denominators from a procedural point of view, while task **5:9 Walkathon** involves such a calculation in context. Task **5:10 Number System, Number Line** part (1) is an unknown-addend problem that includes both a fraction and a decimal.

3:6



3:6 Using what you know about fractions, decide which is greater, $\frac{1}{73}$ or $\frac{1}{41}$. Tell how you decided.

3:7

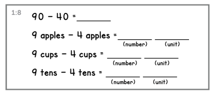


3:7 Here is a list of numbers. Where does each number belong on the number line?

$\frac{1}{2}, \frac{1}{4}, 2, \frac{5}{4}, \frac{2}{2}, \frac{3}{2}, \frac{6}{4}, \frac{2}{4}, \frac{3}{1}$

Draw a dot to show the location of each number. Label each dot. The first number in the list has been located for you.

1:8



1:8 90 − 40 = ______

9 apples − 4 apples = ______ (number) ______ (unit)

9 cups − 4 cups = ______ (number) ______ (unit)

9 tens − 4 tens = ______ (number) ______ (unit)

In earlier grades, task **3:6 Unit Fraction Ideas** focuses on these building blocks of fractions, and task **3:7 Locating Numbers on a Number Line** involves whole numbers and fractions together on the number line including simple cases of equivalent fractions. Task **1:8 Subtracting Units** involves thinking about subtraction of whole numbers in units of tens, which is analogous to thinking about subtraction of fractions in terms of unit fractions.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:8 Shapes with Given Positions

Teacher Notes

Central math concepts

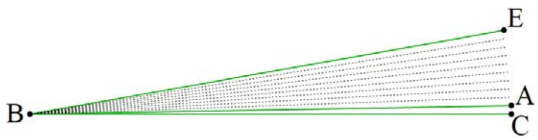
A geometric point with zero size is an abstraction, since any physical drawing of a point, no matter how sharp the pencil we use to draw it, has nonzero diameter. Therefore a point is an idea, and understanding that idea necessarily involves the imagination. Similarly, a geometric line or ray is infinite in extent, even though no physical drawing of a line could be infinitely long or perfectly straight. And the plane to which all these figures belong is no less imaginary an object, given its perfect flatness and its lack of edges. Indeed the infinite sizes of lines and planes, and their constitution as infinitely dense sets of points, are what make them suitable as *number* lines and *coordinate* planes—since the real numbers themselves are both infinite and infinitely dense.

Fortunately, it isn't necessary to draw infinitely long or infinitely straight lines in order to reason about them. Rather, geometric reasoning proceeds on the basis of creating, analyzing, and discussing diagrams that depict geometric objects and relationships. These diagrams have to be true enough for the purpose, but they will usually have conventional features that aren't to be taken literally, such as the thickness of the red point of origin of ray *R* in task 4:8, or the red arrowheads on line *L*. (Lines don't have arrowheads, nor do they have endpoints to which an arrowhead could be attached.) New learners may need to discuss the ways in which a diagram does and doesn't depict geometric "reality."

Geometric measurement advances in grade 4 with angle measurement. Angle measurement resembles other kinds of measurement in its reliance on a unit (CCSS 4.MD.C.5):

- An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and can be used to measure angles.
- An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.

For example, the figure shows a one-degree angle ABC being used as a unit to measure a ten-degree angle EBC.



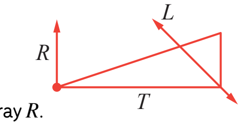
As shown in the table, ideas of angle measure are parallel with ideas of area measure from grade 3, when students learn to recognize area as a measurable attribute of plane figures and to understand concepts of area measurement (CCSS 3.MD.C.5).

4:8 L is a line, R is a ray, and T is a triangle. True or false:

(1) Line L is a line of symmetry for triangle T.

(2) Line L intersects ray R.

(3) Triangle T has two angles measuring less than 90 degrees.



Answer

(1) False. **(2)** True. **(b)** True.

Click here for a student-facing version of the task.

Refer to the Standards

4.MD.C, 4.G.A; MP.5, MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- Students can choose whether to use a protractor to answer part (3).
- The given information that "*L* is a line" and "*R* is a ray" means that these objects satisfy their geometric definitions; in particular, they aren't finite in length.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:8? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:8? In what specific ways do they differ from 4:8?

Area Measure	Angle Measure
A square with side length 1 unit, called “a unit square,” is said to have “one square unit” of area, and can be used to measure area.	An angle that turns through $\frac{1}{360}$ of a circle is called a “one-degree angle,” and can be used to measure angles.
A plane figure that can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units.	An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.

Volume measurement provides additional parallels to angle, area, and length measurement; see CCSS 5.MD.C.3 and the Teacher Notes for task **5:3 Neighborhood Garden**.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: using visual and spatial reasoning; and making measurements on a diagram.

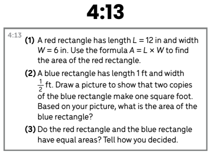
Extending the task

How might students drive the conversation further?

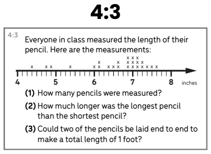
- Students could trace the figure on a blank sheet of paper and use a straightedge to extend line *L* and ray *R* showing their intersection point.
- Students could discuss whether triangle *T* has any line of symmetry.

Related Math Milestones tasks

4:13


4:13
(1) A red rectangle has length $L = 12$ in and width $W = 6$ in. Use the formula $A = L \times W$ to find the area of the red rectangle.
(2) A blue rectangle has length 1 ft and width $\frac{1}{2}$ ft. Draw a picture to show that two copies of the blue rectangle make one square foot. Based on your picture, what is the area of the blue rectangle?
(3) Do the red rectangle and the blue rectangle have equal areas? Tell how you decided.

4:3


4:3
Everyone in class measured the length of their pencil. Here are the measurements:

(1) How many pencils were measured?
(2) How much longer was the longest pencil than the shortest pencil?
(3) Could two of the pencils be laid end to end to make a total length of 1 foot?

Task **4:13 Area Units** involves geometric measurement in relation to area units of differing sizes. The arrowhead on the number line in task **4:3 Pencil Data** has similar purpose to the arrowheads on line *L* and ray *R* in task 4:8.

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:8 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

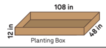
* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

5:8

5:8 A *scalene triangle* is a triangle in which the sides all have different lengths. Thinking about this, Alana decided there should also be a name for quadrilaterals in which the sides all have different lengths. She said, "I'll name them after myself." She defined an *alana-gon* to be a quadrilateral in which the four sides all have different lengths. **(1)** Draw an example of an alana-gon. **(2)** True or false: **(a)** All squares are alana-gons. **(b)** No trapezoids are alana-gons.

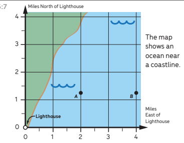
5:3

5:3 A neighborhood garden will have 6 wooden planting boxes. Every box will have the same shape (see diagram). Soil can be bought by the truckload; a truckload is 54 ft^3 of soil. How many truckloads of soil will fill all of the boxes?



5:7

5:7



Shipwrecks are at locations *A* $(2, 1\frac{1}{4})$ and *B* $(4, 1\frac{1}{4})$. Shipwrecks are also at locations *C* $(4, 3\frac{1}{2})$ and *D* $(2, 3\frac{1}{2})$. **(1)** Mark *C* and *D* on the map and shade rectangle *ABCD*. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

5:11

5:11 Juliet said, "I'm thinking of a rectangle. Its area is 1 square unit. Its perimeter is more than 1 million units."

(1) Is Juliet thinking of something possible or impossible? Use math to decide for sure.

(2) Explain your reasoning to your classmates. Revise your explanation based on suggestions from your classmates.

6:12

6:12 **(1)** What is the area of the triangle in the coordinate plane with vertices (1, 2), (−5, 2), and (−8, 9)? **(2)** How does the area change if we change the third vertex to (−3, 9)?

In later grades, task **5:8 Alana's New Shape Category** involves classification and hierarchy of shapes. Tasks **5:3 Neighborhood Garden**, **5:7 Shipwrecks**, and **5:11 Juliet's Rectangle** involve geometric measurement. Task **6:12 Coordinate Triangle** places a geometric figure in the coordinate plane.

3:8

3:8

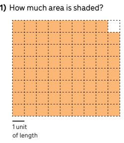
(1) Name two attributes that are shared by triangles and squares.

(2) Name a category of shapes that includes triangles and squares and also includes other shapes that have both of the attributes you named.

3:3

3:3

(1) How much area is shaded?



(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: __________ Width: __________

In earlier grades, task **3:8 Shape Attributes and Categories** involves defining attributes and classification of shapes. Task **:3 Length and Area Quantities** involves concepts of area measurement.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:9 Fitness Day

Teacher Notes

Central math concepts

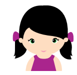
Based on interviews with middle-grades students,[†] education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract…; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "…if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

Especially when a word problem involves fractional quantities, the only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

In particular, task 4:9 involves addition and subtraction and the relationship between them. The mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. (One might paraphrase this statement by saying that "given a total and one part, subtraction finds the other part.") Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

4:9 In gym it was fitness day. Students ran laps around the gym.



Leslie

I ran $1\frac{2}{3}$ more laps than Catherine.

I ran $6\frac{1}{3}$ laps.

How many laps did Catherine run?

Answer

$4\frac{2}{3}$ laps or $\frac{14}{3}$ laps.

Click here for a student-facing version of the task.

Refer to the Standards

4.NF.B.3d; MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

Word problems involving Compare situations can sometimes consist of complex text. Therefore, task 4:9 presents the given information in the form of a monologue by Leslie. This could also invite an approach of having students convey the task to each other by reciting the monologue.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:9? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:9? In what specific ways do they differ from 4:9?

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between addition and subtraction can play out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:[‡]

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems.

In particular, the situation type in task 4:9 is called "Compare with Smaller Unknown." It is a Compare situation because Leslie is using subtraction to compare her distance with Catherine's distance; and more specifically, the situation is "Compare with Smaller Unknown" because the initially unknown quantity is how far Catherine walked (and Catherine walked the shorter distance). During the primary grades, students work with all situation types and all variations in the known and unknown quantities, with quantities given as whole numbers. In the upper-elementary grades, these understandings of addition and subtraction are extended into working with fractional quantities. Although the algorithms for performing calculations with fractions are different from those for performing base-ten calculations with whole numbers, the underlying meanings and uses of addition and subtraction are the same regardless of whether the numbers involved are whole numbers, fractions, decimals, or even variables.

The difference $6\frac{1}{3} - 1\frac{2}{3}$ could be calculated in many ways. Such a calculation need not be approached as an algorithmic process. For example, a student could express both mixed-number addends as fractions, rewriting the problem as $\frac{19}{3} - \frac{5}{3}$. Then, thinking of $\frac{1}{3}$ as a unit of *thirds*, the required difference is 19 thirds - 5 thirds = 14 thirds, for a result of $\frac{14}{3}$. Or, a student could rewrite 6 as 4 + 2, recasting the problem as $4 + 2 + \frac{1}{3} - 1\frac{2}{3}$ and then seeing an opportunity to group as $4 + \frac{1}{3} + (2 - 1\frac{2}{3})$. Now recognizing that $2 - 1\frac{2}{3} = \frac{1}{3}$ yields the result $4 + \frac{1}{3} + \frac{1}{3} = 4\frac{2}{3}$. And still other approaches could be used.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: subtracting fractions with equal denominators; working with mixed numbers; using number sense of fraction size; and representing addition and subtraction on a number line.

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:9 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

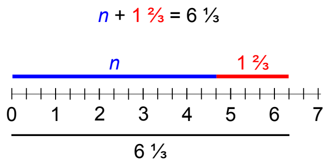
‡ See Table 2, p. 9 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Extending the task

How might students drive the conversation further?

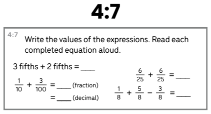
- Students could be asked to write an equation model for the situation, such as $n + 1\frac{2}{3} = 6\frac{1}{3}$. The equation could be represented on a number line, as shown in the figure.



- Students could check the reasonableness of their answers, or predict the approximate value of the answer, by estimating that the answer will be close to 4 laps since $6\frac{1}{3}$ is close to 6 and $1\frac{2}{3}$ is close to 2.
- Students could add a third person to the story who outdoes both Leslie and Catherine. Students could create new dialogue and ask a new question. For example, new dialogue could say, "Oh yeah? I ran 10 laps," and a new question could be, "How many more laps did the new person run than Leslie?"

Related Math Milestones tasks

4:7


4:7 Write the values of the expressions. Read each completed equation aloud.

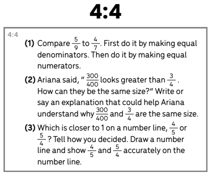
3 fifths + 2 fifths = ____

$\frac{6}{25} + \frac{6}{25} =$ ____

$\frac{1}{10} + \frac{3}{100} =$ ____ (fraction)
$=$ ____ (decimal)

$\frac{1}{8} + \frac{5}{8} - \frac{3}{8} =$ ____

4:4


4:4

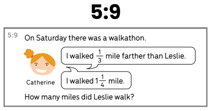
(1) Compare $\frac{5}{9}$ to $\frac{4}{7}$. First do it by making equal denominators. Then do it by making equal numerators.

(2) Ariana said, "$\frac{300}{400}$ looks greater than $\frac{3}{4}$. How can they be the same size?" Write or say an explanation that could help Ariana understand why $\frac{300}{400}$ and $\frac{3}{4}$ are the same size.

(3) Which is closer to 1 on a number line, $\frac{4}{5}$ or $\frac{5}{4}$? Tell how you decided. Draw a number line and show $\frac{4}{5}$ and $\frac{5}{4}$ accurately on the number line.

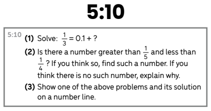
Task **4:7 Fraction Sums and Differences** is a non-contextual task about finding sums and differences of fractions with equal denominators. Task **4:4 Comparing Fractions with Equivalence** (part 3) involves fraction comparisons and distances on a number line.

5:9


5:9 On Saturday there was a walkathon.

How many miles did Leslie walk?

5:10

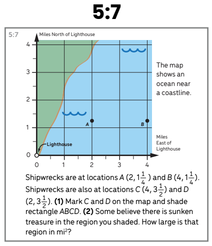

5:10

(1) Solve: $\frac{1}{3} = 0.1 + ?$

(2) Is there a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$? If you think so, find such a number. If you think there is no such number, explain why.

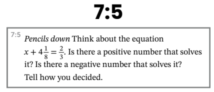
(3) Show one of the above problems and its solution on a number line.

5:7



Shipwrecks are at locations A $(2, 1\frac{1}{4})$ and B $(4, 1\frac{1}{4})$. Shipwrecks are also at locations C $(4, 3\frac{1}{2})$ and D $(2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

7:5


7:5 *Pencils down* Think about the equation $x + 4\frac{1}{8} = \frac{2}{3}$. Is there a positive number that solves it? Is there a negative number that solves it? Tell how you decided.

In later grades, task **5:9 Walkathon** is a word problem of situation type "Compare with Smaller Unknown" in which the calculation is similar to the one in task 4:9, except with unequal denominators. Task **5:10 Number System, Number Line** (part (1)) features an unknown addend problem involving both a fraction and a decimal. Task **5:7 Shipwrecks** involves finding the difference between $3\frac{1}{2}$ and $1\frac{1}{4}$. Task **7:5 Is There a Solution? (Addition)** involves an equation for an unknown addend in the rational numbers.

In earlier grades, see the [Map of Addition and Subtraction Situations in K–2 Math Milestones](#).

4:9 Fitness Day

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:10 Calculating Products and Quotients

Teacher Notes

Central math concepts

Task 4:10 focuses on procedures. For the mental calculations in the task, students can use place value, properties of operations, and (in the case of 480 ÷ 8) the relationship between multiplication and division as computation strategies. For the pencil-and-paper calculations in the task, there are various possibilities for efficient, accurate, and generalizable algorithms that handle the given problems.

> **Computation algorithm**. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.
>
> **Computation strategy**. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Computation strategies and computation algorithms are usefully distinguished (CCSS Glossary). Strategies are "purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another." Mental calculation often uses such strategies. For example, we could calculate 5 × 19 mentally by thinking of 5 × 20 - 5, then thinking of 5 × 20 as 5 × 2 × 10, which is 10 × 10 or 100. Thus, the result is 100 - 5 = 95. Alternatively, we could think of 5 × 19 as 50 + 45, obtaining 95 that way. Flexible procedural fluency based on place value, properties of operations, relationships between operations, and ideas of equivalence is valuable both in practical terms and also as a way of doing arithmetic that prefigures algebra. In algebra, strategies are the most common way of working.

Algorithms are inflexible by definition. One step follows another in the prescribed order. Some algorithms are also much simpler to execute than others. The standard multi-digit addition algorithm is less complex than the standard multi-digit subtraction algorithm. The standard algorithm for dividing fractions is less complex than the standard algorithm for dividing multi-digit numbers.

Sometimes even an efficient general-purpose algorithm wouldn't be an efficient approach to a particular instance of a calculation, as in a subtraction problem like 4,003 - 8. On the other hand, when faced with a calculation there may be times when we don't find ourselves readily inventing a flexible mental procedure on the spot, so it's valuable to know and be proficient with an algorithm.

The grade 4 standards (CCSS 4.NBT.B.5, 6) do not set an expectation of fluency for multi-digit multiplication and division; in this grade, multiplying and dividing multi-digit numbers is a substantially conceptual process. (See the Teacher Notes for task **4:2 Multi-Digit Division Concepts**.) However, practice with grade 4 products and quotients can promote confidence while offering opportunities to debug procedures, reinforce ideas, and strengthen recall of single-digit products. (See the Teacher Notes for task **3:12 Products of Single-Digit Numbers** and the Teacher

---

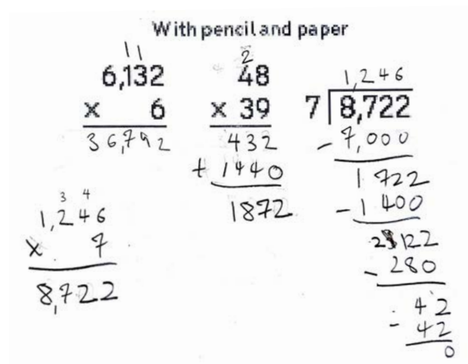
4:10 Write the values of the products and quotients. Check the quotients by multiplying.

Mentally
40 × 20
30 × 11
12 × 60
5 × 19
480 ÷ 8

With pencil and paper
6,132 × 6   48 × 39   7)8,722

Answer

For the written calculations, the answers are 36,792; 1,872; and 1,246. For samples of written work, including the multiplication check 1246 × 7 for the problem 8722 ÷ 7, see the examples shown. Students might use different algorithms than the ones shown in the examples.



For the mental calculations, results are shown in **this color**. Some potential thought processes leading to the results (*not the only possible thought processes*) are shown in **this color**. Note that for the mental calculations, it is fine if students manage the mental load by using writing to jot down intermediate results along the way.

40 × 20 = **(4 × 2) × 100 = 8 × 100 = 800**

30 × 11 = **(3 × 11) × 10 = 33 × 10 = 330**

12 × 60 = **10×60 + 2×60 = 600 + 120 = 720**

5 × 19 = **5×20 - 5 = 100 - 5 = 95**

480 ÷ 8 = **(48 ÷ 8) × 10 = 6 × 10 = 60**

Check: 60 × 8 = **(6 × 8) × 10 = 48 × 10 = 480**

Click here for a student-facing version of the task.

Notes for task **3:13 Fluency within the Multiplication Table**.)

Fluency in particular with multiplying a multi-digit number by a single-digit number, as in the problem 4087 × 3 or the problem 4087 × 5, could help to prepare for fluently multiplying multi-digit numbers by multi digit numbers in grade 5, because of the way a grade 5 calculation like 4087 × 53 involves a sequence of such operations (see figure).

$$\begin{array}{r} 4087 \\ \times \quad 53 \\ \hline 12261 \\ 204350 \\ \hline 216611 \end{array}$$

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value; single-digit products and related quotients; using the commutative, associative, and distributive properties; and the relationship between multiplication and division.

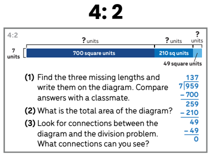
Extending the task

How might students drive the conversation further?

- Students could make sense of their answers by making estimates of the values; for example, 48 × 39 should be somewhat less than 50 × 40 = 2,000.
- Students could help each other debug mistakes that may arise.

Related Math Milestones tasks

4: 2



4:6

4:6 Grandpa took a jar of pennies to the bank. He said, "I'd like nickels for this, please." The bank teller poured the pennies into a counting machine. "Eighty-seven dollars and forty-two cents," said the teller. **(1)** How many nickels did Grandpa get? **(2)** Check your answer with an estimate.

4:11

4:11 A cook in the school kitchen uses 6 oz of cheese to make a pizza. The kitchen has 45 lb of cheese. How many pizzas will that make?

4:14

4:14 540,909 + 87,808 – 5,864 + 2,556 = ?

Task **4:2 Multi-Digit Division Concepts** involves multi-digit division from a conceptual point of view. Grade-level multiplication and division calculations could be performed in context for tasks **4:6 Jar of Pennies** and **4:11 School Kitchen**. Task **4:14 Fluency with Multi-Digit Sums and Differences** marks the culmination of the progression for multi-digit addition and subtraction.

5:5

5:5 Write the requested values.

4087 × 53 = ?	$\frac{1}{10} \div 10 = ?$	0.4 × 0.9 = ?
246 × 914 = ?	$\frac{7}{8} \times \frac{5}{3} = ?$	0.75 ÷ 0.01 = ?
9744 ÷ 12 = ?	8 × ? = 73	0.63 ÷ 0.3 = ?
1461 ÷ 6 = ?	$3 \div \frac{1}{8} = ?$	0.86 + 0.4 = ?
4 – (8 – 4) = ?	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	0.72 – 0.17 = ?
	$\frac{1}{3} \div (6 \times 5) = ?$	0.02 + 0.2 = ?
		0.8 – 0.55 = ?
		637 – 1.31 = ?

6:14

6:14 *Pencil and paper.* **(1)** 81.53 ÷ 3.1 = ? **(2)** $\frac{7}{8} \div \frac{2}{3} = ?$ **(3)** Check both of your answers by multiplying.

In later grades, task **5:5 Calculating** involves grade-level procedures for addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals. Task **6:14 Dividing Decimals and Fractions** marks the culmination of algorithmic procedures with general fraction and decimal division.

Refer to the Standards

4.NBT.B; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

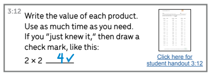
- For the mental calculations, it is fine if students manage the mental load by using writing to jot down intermediate results along the way.
- The task does not require students to show their work, but looking at students' steps can show where they may have made a careless mistake.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:10? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 4:10? In what specific ways do they differ from 4:10?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:10 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

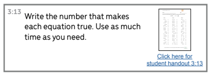
3:12


3:12 Write the value of each product. Use as much time as you need. If you "just knew it," then draw a check mark, like this:

2 × 2 4✓

Click here for student handout 3:12

3:13


3:13 Write the number that makes each equation true. Use as much time as you need.

Click here for student handout 3:13

In earlier grades, tasks **3:12 Products of Single-Digit Numbers** and **3:13 Fluency within the Multiplication Table** concern single-digit products and related quotients, which are building blocks for multi-digit multiplication and division algorithms.

4:10 Calculating Products and Quotients

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:11 School Kitchen

Teacher Notes

Math Milestones™

Central math concepts

By high school, converting units is a proceduralized skill—especially as applied in science coursework and laboratory work. In the elementary and middle grades, however, "converting units" is a thoughtful process of multiplicative reasoning that involves recognizing how many times-as-much a certain unit is than another unit. This reasoning might be fairly abstract or fairly concrete depending on the kind of quantity under consideration, ranging from base quantities such as length, mass, and time to derived quantities such as area, volume, speed, and force.

Grade 4 problems involve expressing measurements in a larger unit in terms of a smaller unit (CCSS 4.MD.A.1), within a single system of measurement. In such cases, the larger unit will be a whole-number multiple of the smaller unit. For task 4:11 in particular, weight is measured in both pounds and ounces. As measured by a calibrated scale, 1 pound weighs 16 times as much as 1 ounce. Therefore, a quantity of 45 pounds weighs 45 × 16 ounces. This is not the application of a remembered rule ("To convert pounds to ounces, multiply by 16") but rather the application of multiplication thinking to a remembered fact (1 pound is 16 ounces).

Unit thinking is prevalent throughout arithmetic, not just in the measurement domain. The idea of a unit is a coherent and unifying theme in school mathematics.[†] In task 4:11 for example, 6 ounces of cheese could be viewed as a unit, "1 pizza's worth of cheese." From this perspective, a final division step solves task 4:11 by measuring the kitchen's cheese supply in units of pizzas.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: multiplying by a unit fraction; applying ideas of times-as-much in context; and basing multiplicative reasoning on math diagrams.

Extending the task

How might students drive the conversation further?

- Knowing that 45 pounds of cheese will make 120 pizzas with 6 ounces of cheese on each pizza, students could consider such questions as
  - What if the kitchen had 90 pounds of cheese instead of 45 pounds? How many pizzas would that make?
  - What if there were 3 ounces of cheese on each pizza instead of 6 ounces? How many pizzas would 45 pounds of cheese make in that case?
- Intuitive answers to questions like these could be checked by calculation.

> 4:11 A cook in the school kitchen uses 6 oz of cheese to make a pizza. The kitchen has 45 lb of cheese. How many pizzas will that make?

Answer

120.

Click here for a student-facing version of the task.

Refer to the Standards

4.MD.A.2, 4.NBT.B.5; MP.1, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

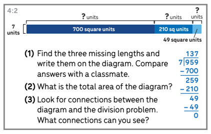
The numbers in the problem are such that they provide opportunities for grade-level procedures such as calculating 45 × 16 and 720 ÷ 6.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:11? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:11? In what specific ways do they differ from 4:11?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:11 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

Related Math Milestones tasks

4:2


4:2

? units ? units ? units

7 units

700 square units 210 sq units

49 square units

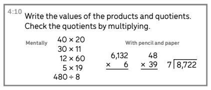
(1) Find the three missing lengths and write them on the diagram. Compare answers with a classmate.

(2) What is the total area of the diagram?

(3) Look for connections between the diagram and the division problem. What connections can you see?

$$\begin{array}{r} 137 \\ 7\overline{)959} \\ -700 \\ \hline 259 \\ -210 \\ \hline 49 \\ -49 \\ \hline 0 \end{array}$$

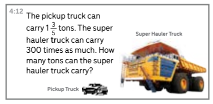
4:10


4:10 Write the values of the products and quotients. Check the quotients by multiplying.

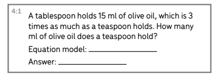
Mentally: 40×20, 30×11, 12×60, 5×19, $480 \div 8$

With pencil and paper: $6{,}132 \times 6$, 48×39, $7\overline{)8{,}722}$

4:12


4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

4:1

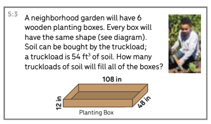

4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ____________

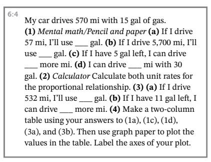
Answer: ____________

Calculations like those involved in task 4:11 are included from a conceptual point of view in task **4:2 Multi-Digit Division Concepts** and from a procedural point of view in task **4:10 Calculating Products and Quotients**. Task **4:12 Super Hauler Truck** is a word problem involving multiplicative comparison (situation type Compare with Larger Unknown), and task **4:1 A Tablespoon of Oil** is a word problem involving multiplicative comparison (situation type Compare with Smaller Unknown) that happens to involve quantities in common use as kitchen measurements.

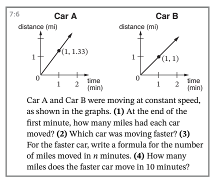
5:3


5:3 A neighborhood garden will have 6 wooden planting boxes. Every box will have the same shape (see diagram). Soil can be bought by the truckload; a truckload is 54 ft³ of soil. How many truckloads of soil will fill all of the boxes?

6:4

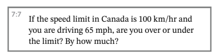

6:4 My car drives 570 mi with 15 gal of gas. **(1)** *Mental math/Pencil and paper* **(a)** If I drive 57 mi, I'll use ___ gal. **(b)** If I drive 5,700 mi, I'll use ___ gal. **(c)** If I have 5 gal left, I can drive ___ more mi. **(d)** I can drive ___ mi with 30 gal. **(2)** *Calculator* Calculate both unit rates for the proportional relationship. **(3)** **(a)** If I drive 532 mi, I'll use ___ gal. **(b)** If I have 11 gal left, I can drive ___ more mi. **(4)** Make a two-column table using your answers to (1a), (1c), (1d), (3a), and (3b). Then use graph paper to plot the values in the table. Label the axes of your plot.

7:6


7:6

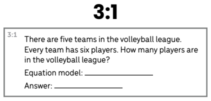
Car A and Car B were moving at constant speed, as shown in the graphs. **(1)** At the end of the first minute, how many miles had each car moved? **(2)** Which car was moving faster? **(3)** For the faster car, write a formula for the number of miles moved in *n* minutes. **(4)** How many miles does the faster car move in 10 minutes?

7:7


7:7 If the speed limit in Canada is 100 km/hr and you are driving 65 mph, are you over or under the limit? By how much?

In later grades, task **5:3 Neighborhood Garden** involves two different units of measure for volume, with the larger unit being more convenient for solving the problem. Proportional relationships in tasks such as **6:4 Gas Mileage** involve derived quantities, and rates are compared in tasks **7:6 Car A and Car B** and **7:7 Speed Limit**.

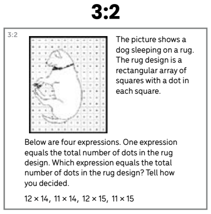
3:1


3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?

Equation model: ____________

Answer: ____________

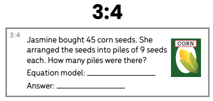
3:2


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

12×14, 11×14, 12×15, 11×15

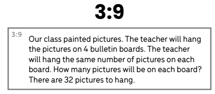
3:4


3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?

Equation model: ____________

Answer: ____________

3:9


3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

In earlier grades, tasks such as **3:1 Volleyball Players**, **3:2 Hidden Rug Design**, **3:4 Corn Seeds**, and **3:9 Bulletin Board Pictures** focus on the equal-groups concept of multiplication that is the precursor of times-as-much thinking.

† See Zimba (2013), "Units, a Unifying Theme in Measurement, Fractions, and Base Ten."

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Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:12 Super Hauler Truck

Teacher Notes

Central math concepts

Task 4:12 involves multiplicative comparison, which represents an advance on grade 3 multiplicative thinking. In multiplication and division word problems in grade 3, students multiplied whole numbers to find the total number of objects when the objects were grouped equally or were arrayed in rows and columns (including cases where the objects in rows and columns were square units), and students divided whole numbers to find an unknown factor in such situations (unknown group size, unknown number of groups, or unknown length measure). In grade 4, students will apply and extend their thinking about multiplication and division to solve problems of multiplicative comparison.[†]

Multiplicative comparison problems involve the idea of times-as-many/times-as-much, which is the idea that the product $A \times B$ refers to a quantity that is A times as many/times as much as the quantity B.

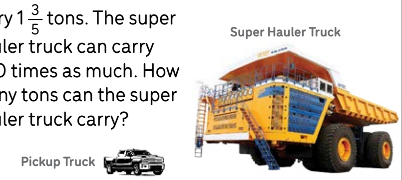
The conceptual or perhaps linguistic difference between times-as-many and times-as-much is that times-as-*many* applies most directly to discrete objects, like bowling balls or boats. Times-as-*much* applies most directly to substances that can be measured out and repeatedly subdivided, like a quantity of fluid or an interval of time. Often, the measures of these substances are fractions. As a conceptual step beyond the idea of equal-groups, the idea of times-as-many/times-as-much may be an important step closer to the grade 5 idea of multiplication as a scaling operation that magnifies or shrinks a quantity.

Extending multiplication and division from whole numbers—and from whole-number-specific mental models like equal groups—to fractions and to the mental models that accommodate fractions, like times-as-much, is perhaps the most important mathematical progression of the upper-elementary grades. The need to support students in making that conceptual evolution raises important questions: What aspects of earlier thinking about multiplying whole numbers will remain helpful when making sense of a product involving fractions? What new ways of thinking will be helpful? And what mathematical representations introduced during whole-number multiplication work are best suited to supporting the transition from multiplying and dividing whole numbers to multiplying and dividing all numbers?

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: mental math such as $300 \div 5$ or 8×300; and working with mixed numbers and fractions.

4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?



Answer

480 tons.

Click here for a student-facing version of the task.

Refer to the Standards

4.NF.B.4c, 4.OA.A.2; MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

- The targeted mathematics in the task is multiplication, not addition. The multiplier of 300 in the task is intentionally large, so as to invite multiplicative thinking.
- Super hauler trucks can, in fact, carry such large payloads; see Haul Truck on Wikipedia.
- In the image for the task, the pickup truck and super hauler truck are shown to actual scale.

Extending the task

How might students drive the conversation further?

- Students could ask and answer the question, "How many pounds can each truck carry?"
- Students could use the image in the task (which accurately shows the relative sizes of the trucks) to estimate how many times taller or how many times longer the super hauler truck is compared to the pickup truck.

Related Math Milestones tasks

4:1

4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ____________

Answer: ____________

4:5

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} = \underset{(a)}{\_\_\_\_}$

$6 \times \frac{4}{7} = \underset{(b)}{\_\_\_\_}$

$86 \times \frac{1}{86} = \underset{(c)}{\_\_\_\_}$

$6 \times \frac{8}{2} = \underset{(d)}{\_\_\_\_}$

$9 \times \frac{1}{9} = \underset{(e)}{\_\_\_\_}$

$9 \times \frac{2}{9} = \underset{(f)}{\_\_\_\_}$

(1g) Which answer is twice as much as the answer for (e)?

(1h) Which answer is six times as much as the answer for (a)?

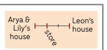
(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

Task **4:1 A Tablespoon of Oil** is a word problem involving multiplicative comparison (situation type Compare with Smaller Unknown). Task **4:5 Fraction Products and Properties** concentrates on the concepts of multiplying a fraction by a whole number.

5:6

5:6 **(1)** Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?



5:13

5:13 In a snack shop there is a frozen yogurt machine. When there is 3 *l* of frozen yogurt in the machine, the machine is $\frac{1}{3}$ full. How much frozen yogurt is in the machine when it is $\frac{1}{4}$ full?

6:1

6:1 $\frac{2}{3}$ of a charging cord is $\frac{1}{2}$ meter long. How long is the charging cord? (Answer in meters.)

6:9

6:9 How much of a $\frac{3}{4}$-ton truckload is $\frac{2}{3}$ ton of gravel?

In later grades, task **5:6 Corner Store** involves products of unit fractions and products of other fractions in context, and task **5:13 Frozen Yogurt Machine** involves both multiplying and dividing with fractions. Tasks **6:1 Charging Cord** and **6:9 Truckload of Gravel** are word problems that involve finding an unknown factor in a fraction product.

3:2

3:2

The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

$12 \times 14,\ 11 \times 14,\ 12 \times 15,\ 11 \times 15$

3:1

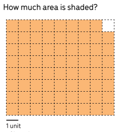
3:1 There are five teams in the volleyball league. Every team has six players. How many players are in the volleyball league?

Equation model: ____________

Answer: ____________

3:3

3:3 **(1)** How much area is shaded?



(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ____________ Width: ____________

3:4

3:4 Jasmine bought 45 corn seeds. She arranged the seeds into piles of 9 seeds each. How many piles were there?

Equation model: ____________

Answer: ____________

3:9

3:9 Our class painted pictures. The teacher will hang the pictures on 4 bulletin boards. The teacher will hang the same number of pictures on each board. How many pictures will be on each board? There are 32 pictures to hang.

In earlier grades, task **3:2 Hidden Rug Design** centers on the equal-groups concept of multiplication in an array context, while tasks **3:1 Volleyball Players**, **3:4 Corn Seeds**, and **3:9 Bulletin Board Pictures** are word problems centered on equal-groups concepts of multiplication and/or division. Multiplication is useful in task **3:3 Length and Area Quantities**.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:12? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:12? In what specific ways do they differ from 4:12?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:12 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† See Table 3, p. 23 of *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

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Language

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Identity, Agency, and Belonging

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Math Milestones™

Central math concepts

To work successfully with measured quantities, and also to reason with place value and fractions, students must attend to units.[†] For example, suppose we have a roll of 10 **yards** of packing tape and another roll of 30 **feet** of packing tape. Then the total length of packing tape isn't 10 + 30 = 40 yards. Nor is the total length 10 + 30 = 40 feet.[‡]

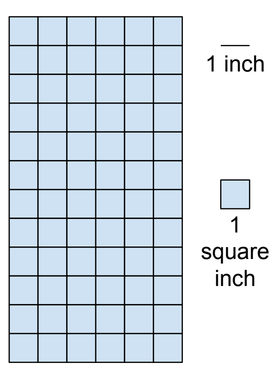
If we measure the same quantity using different units, then the numerical value of the measure will be different, even though the quantity itself is the same. For example, the length of a snow leopard's tail might have the value 36 when measured in units of inches, and a value of 3 when measured in units of feet. But it's the same tail. An inverse multiplicative relationship is present here: a foot is 12 times longer than an inch, so it takes 12 times as many inches as feet to measure the same length.

In grades K–2, students work with length units and concepts of length measurement (CCSS 1.MD.A.2). In kindergarten, students make non-numerical comparisons of length and other measurable quantities (CCSS K.MD.A). In grade 1, using objects as length units, students learn that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. In grade 2, students extend these ideas to abstract length units and relate length measurement to addition, subtraction, and the number line (CCSS 2.MD.A).

After studying length as a measurable quantity in the primary grades, students learn to recognize area as a measurable attribute of plane figures. Area is the amount of two-dimensional surface a figure contains. Consistent with this idea, congruent figures are assumed to enclose equal areas.

Students also understand the concepts involved in measuring area (CCSS 3.MD.C.5):

- **A unit of measure for area:** An area unit is built from a chosen length unit. Given a length unit, a square with side length equal to 1 unit, called "a unit square," is said to have "one square unit" of area.
- **Quantifying area:** A plane figure that can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.



For example, if a rectangle has length 12 inches and width 6 inches, then 72 square inches will cover the rectangle with no gaps or overlaps (see figure).

In task 4:13, if the length unit is 1 inch, then applying the formula $A = L \times W$ to the red rectangle leads to the calculation $A = 6 \times 12 = 72$. If the length unit is 1 foot, then applying the formula $A = L \times W$ to the blue rectangle leads to the calculation $A = 1 \times \frac{1}{2} = \frac{1}{2}$. The numbers 72 and $\frac{1}{2}$ are unequal,

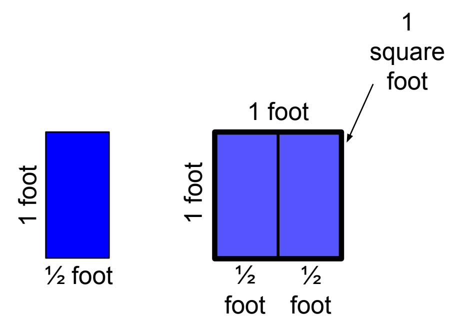
4:13 **(1)** A red rectangle has length $L = 12$ in and width $W = 6$ in. Use the formula $A = L \times W$ to find the area of the red rectangle.

(2) A blue rectangle has length 1 ft and width $\frac{1}{2}$ ft. Draw a picture to show that two copies of the blue rectangle make one square foot. Based on your picture, what is the area of the blue rectangle?

(3) Do the red rectangle and the blue rectangle have equal areas? Tell how you decided.

Answer

(1) 72 in^2. **(2)** See example picture; the picture shows that the area of the blue rectangle is half of a square foot, or $\frac{1}{2}$ ft^2. **(3)** Yes. Explanations may vary but could include the idea that the red and blue rectangles can be laid atop one another exactly; that 72 square inches can tile half of a square foot; or that 144 square inches can tile one square foot.



Click here for a student-facing version of the task.

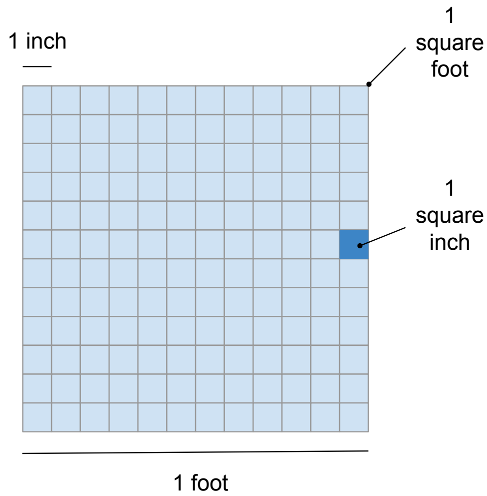
Refer to the Standards

4.MD.A.3; MP.3, MP.5. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

but 72 in^2 and $\frac{1}{2}$ ft^2 refer to equal quantities of area. The number 72 is 144 times as much as $\frac{1}{2}$, and this is a consequence of the fact a square foot has 144 times as much area as a square inch (see figure).



Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: spatially structuring a rectangle into an array; multiplying a two-digit number by a single-digit number; using a formula; and working with measurement concepts.

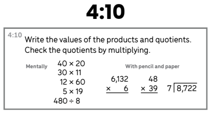
Extending the task

How might students drive the conversation further?

- Students could explore what the area measure of the red rectangle would be if a new, smaller unit of area were chosen, such as a square with side length $\frac{1}{4}$ inch. (Compared to their length measures in inches, the length measures of the rectangle when using the smaller unit are both increased by a factor of 4, so the area formula implies a total increase in the area by a factor of 16. From another perspective, each square inch contains 16 of the new smaller units, so the number of the smaller units is 16 times greater.)
- Students could discuss the situation described in "Central math concepts" in which we have one roll of 10 yards of packing tape and another roll of 30 feet of packing tape. How would we assign a numerical measure to the total length of packing tape?

Related Math Milestones tasks

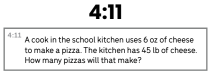
4:10


4:10 Write the values of the products and quotients. Check the quotients by multiplying.

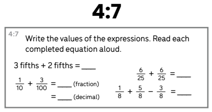
Mentally: 40×20, 30×11, 12×60, 5×19, $480 \div 8$

With pencil and paper: $6{,}132 \times 6$, 48×39, $8{,}722 \div 7$

4:11


4:11 A cook in the school kitchen uses 6 oz of cheese to make a pizza. The kitchen has 45 lb of cheese. How many pizzas will that make?

4:7


4:7 Write the values of the expressions. Read each completed equation aloud.

3 fifths + 2 fifths = ___

$\frac{1}{10} + \frac{3}{100}$ = ___ (fraction) = ___ (decimal)

$\frac{6}{25} + \frac{6}{25}$ = ___

$\frac{1}{8} + \frac{5}{8} - \frac{3}{8}$ = ___

Like part (1) of task 4:13, tasks **4:10 Calculating Products and Quotients** and **4:11 School Kitchen** both involve products outside of the 10 × 10 multiplication table. Like part (2) of task 4:13, task **4:7 Fraction Sums and Differences** involves addition of fractional quantities.

Additional notes on the design of the task

The task does not include a diagram, and that is intentional because thoughtfully building the diagrams is part of the task.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:13? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 4:13? In what specific ways do they differ from 4:13?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:13 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Even adding apples and oranges requires introducing a new unit, such as "fruit." See Teacher Notes for task **K:14 Animals from Land and Sea**. Brief observations and examples of the role of units throughout arithmetic can be found in "Units, a Unifying Idea in Measurement, Fractions, and Base Ten" (blog post by Jason Zimba, 2013).

‡ Is the total amount of packing tape equal to 40 of any length unit? What length unit?

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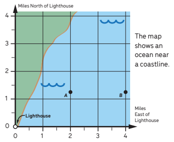
5:7

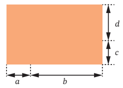
5:7



Shipwrecks are at locations A $(2, 1\frac{1}{4})$ and B $(4, 1\frac{1}{4})$. Shipwrecks are also at locations C $(4, 3\frac{1}{2})$ and D $(2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

6:11

6:11 The diagram shows a rectangle. The variables a, b, c, and d are lengths in meters.



(1) Using the variables, write three different expressions for the area of the rectangle. **(2)** Choose two of your expressions and show that they are equivalent by applying properties of operations. **(3)** State the property or properties you used.

6:12

6:12 **(1)** What is the area of the triangle in the coordinate plane with vertices $(1, 2)$, $(-5, 2)$, and $(-8, 9)$? **(2)** How does the area change if we change the third vertex to $(-3, 9)$?

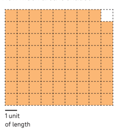
7:7

7:7 If the speed limit in Canada is 100 km/hr and you are driving 65 mph, are you over or under the limit? By how much?

In later grades, task **5:7 Shipwrecks** involves rectangle area in context, for a rectangle with fractional dimensions, and task **6:11 Area Expressions** involves rectangle area in a case where the lengths are variables rather than numbers. Task **6:12 Coordinate Triangle** involves area measure for a triangle. Task **7:7 Speed Limit** involves comparing two quantities of speed given in different units.

3:3

3:3 **(1)** How much area is shaded?



(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ____________ Width: ____________

In earlier grades, task **3:3 Length and Area Quantities** involves concepts of area measurement.

Math Milestones™

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

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Solution Paths

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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

4:14 Fluency with Multi-Digit Sums and Differences

Teacher Notes

Math Milestones™

Central math concepts

Task 4:14 focuses on fluency with procedures and, in particular, fluency with the standard algorithms for multi-digit addition and subtraction. In these algorithms, one sets up a vertical tableau and works from right to left, with the result appearing immediately below the horizontal line.

$$\begin{array}{r} 540909 \\ 87808 \\ +\quad 2556 \\ \hline 631273 \end{array} \qquad \begin{array}{r} 631273 \\ -\quad 5864 \\ \hline 625409 \end{array}$$

Algorithms are usefully distinguished from strategies (CCSS Glossary; see figure). Strategies are "purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another." Mental calculation often uses strategies. For example, we could calculate 205 - 9 mentally by replacing the given difference with the equivalent problem 205 - 5 - 4. Flexible procedural fluency based on place value, properties of operations, relationships between operations, and ideas of equivalence is valuable both in practical terms and also as a way of doing arithmetic that prefigures algebra. In algebra, strategies are the most common way of working.

> **Computation algorithm**. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.
>
> **Computation strategy**. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Algorithms are inflexible by definition. One step follows another in the prescribed order. However, there are still some choices in task 4:14, such as whether to organize the work as shown in the figure above, or in another way (such as by adding the first two numbers, then subtracting the second from the sum, then adding the last number to the difference; or such as by adding the first two numbers, then subtracting from that result the difference between 5,864 and 2,556).

The efficiency of the standard multi-digit algorithms comes from the fact that the place value system is uniform from one place to another. Yet sometimes even an efficient general-purpose algorithm wouldn't be an efficient approach to a particular instance of a calculation, as in a subtraction problem like 4,003 - 8. On the other hand, when faced with a calculation, there may be times when we don't find ourselves readily inventing a flexible mental procedure on the spot, so it's valuable to know and be proficient with an algorithm.

An important value in mathematics education is that of being able to solve problems in multiple ways. This brings the pleasures of seeing how a coherent subject holds together, and it allows students to check answers, unify their understanding of concepts, and learn from different ways of thinking that emerge in the classroom community. A parallel but also important outcome of mathematics education is for students to be

> 4:14
> 540,909 + 87,808 – 5,864 + 2,556 = ?

Answer

625,409.

Click here for a student-facing version of the task.

Refer to the Standards

4.NBT.B.4; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

The task does not require students to show their work, but looking at students' steps can show where they may have made a careless mistake.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 4:14? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 4:14? In what specific ways do they differ from 4:14?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 4:14 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

supported in gaining procedural fluency with algorithms for the actually quite small set of recurrent problem types for which an algorithm exists. This small set can be found in the CCSS-M by searching for “algorithm.”

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value concepts; and single-digit sums and differences.

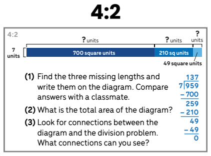
Extending the task

How might students drive the conversation further?

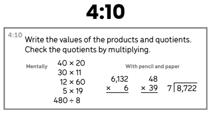
- Checking differences by adding can offer additional procedural practice and reinforce the relationship between addition and subtraction (C - A is the unknown factor in $A + \square = C$).
- Students could make sense of their answers another way by making estimates of the values; for example, the answer in thousands should be reasonably close to 541 + 88 - 3 = 541 + 85 = 626.

Related Math Milestones tasks

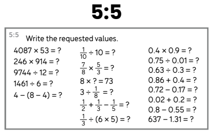
4:2



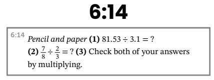
4:10



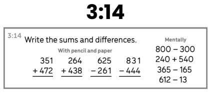
5:5



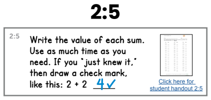
6:14



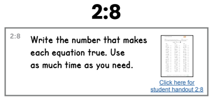
3:14



2:5



2:8



Task **4:2 Multi-Digit Division Concepts** involves another multi-digit algorithm, but at a pre-fluency stage in the learning progression. Task **4:10 Calculating Products and Quotients** involves grade-level procedures with multi-digit multiplication and division.

In later grades, task **5:5 Calculating** continues procedures into larger numbers of digits and into fractions and decimals. Task **6:14 Dividing Decimals and Fractions** marks the culmination of algorithmic procedures with general fraction and decimal division.

In earlier grades, task **3:14 Fluency within 1000 (Add/Subtract)** involves grade-level fluencies for addition and subtraction, while tasks **2:5 Sums of Single-Digit Numbers** and **2:8 Fluency within the Addition Table** involve the single-digit sums and related differences upon which multi-digit addition and subtraction algorithms are built.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?