5:1 Juice Box Mixup

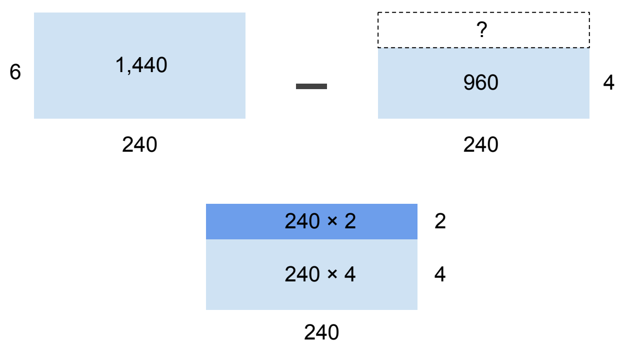
Teacher Notes

Central math concepts

Comparing different approaches to the same problem often reveals important mathematics that would be hard to see within just a single approach. To illustrate the central math concepts in this task, consider the following two distinct approaches to the task:

> *Approach 1*. First, calculate how many juice boxes the school bought: 240 × 6 = 1,440. Next, calculate how many juice boxes the school needed: 240 × 4 = 960. So the school bought 1,440 juice boxes, but the school only needed 960 juice boxes. Therefore, the school bought 1,440 - 960 = 480 extra juice boxes.
>
> *Approach 2*. Because the school only needed to buy 4-packs, every 6-pack bought by the school included 2 juice boxes the school didn't need. Therefore, the school bought 240 × 2 = 480 extra juice boxes.

Both ways of thinking about the problem are good and true, but then that raises a question: how can two such different ways of thinking lead to the same answer? There is a deep connection between the two approaches, and it involves the distributive property. In the figure, an area model is used to illustrate the connection:



These diagrams illustrate the central mathematical relationship between the two approaches, which is that (240 × 6) - (240 × 4) = 240 × 2. Said another way,

$$(240 \times 2) + (240 \times 4) = 240 \times 6.$$

And since 6 = 2 + 4, the previous equation can be written as

$$(240 \times 2) + (240 \times 4) = 240 \times (2 + 4),$$

which is a particular case of the distributive property, $a \times b + a \times c = a \times (b + c)$.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: multiplying a multi-digit number by a single-digit number; using area models to illustrate reasoning that involves multiplication, addition or subtraction, and the

> 5:1 A school needed 240 four-packs of juice boxes for a field trip. However, the school accidentally bought 240 *six-packs* of juice boxes. How many extra juice boxes did the school buy?

Answer

The school bought 480 extra juice boxes.

Click here for a student-facing version of the task.

Refer to the Standards

5.OA.A.2, 5.NBT.B.5; MP.4, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

- The task isn't intended to *differentiate between* students who adopt the strategy 240 × 2 = 480 and students who adopt the strategy 240 x 6 = 1,440; 240 x 4 = 960; 1,440 - 960 = 240. Rather, the task is designed to *foster mathematical conversations* that compare these or other approaches in relation to one another. In other words, the point isn't to decide which strategy is best, but rather to prompt students to identify and discuss meaningful mathematical correspondences between strategies.
- This is not a fluency task; the challenges in the task are subtle and should be met using understanding, not by applying recipes.

distributive property; and viewing expressions like 240 × 6 as objects with structure that can be interpreted—not just as instructions to calculate a resulting value.

Extending the task

How might students drive the conversation further?

- Students could compare different strategies for solving the problem and identify the mathematical properties that ensure the two strategies lead to the same answer.
- Students could create their own versions of the problem with different numbers. For example, the school's mistake might involve substituting 12-packs for 10-packs…why is the final answer the same in this case as it was for the original problem? In discussing variations, keep the conversation close to the central concepts of the original task.

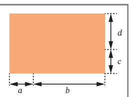
Related Math Milestones tasks

5:14

5:14 Brandon was reading his math book. He saw the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$. He said, "I don't get it—where did the 3 and the $\frac{3}{8}$ come from?" Write an explanation that could answer Brandon's question.

6:11

6:11 The diagram shows a rectangle. The variables a, b, c, and d are lengths in meters.



(1) Using the variables, write three different expressions for the area of the rectangle. **(2)** Choose two of your expressions and show that they are equivalent by applying properties of operations. **(3)** State the property or properties you used.

7:3

7:3 Write each sum as a product with the given factor. *Example:* $8 + 6x = 2 \cdot ?$
Answer: $8 + 6x = 2(4 + 3x)$. **(1)** $6y + 12 = 3 \cdot ?$
(2) $-5w + 35 = (-5) \cdot ?$ **(3)** $4z + 1 = 4 \cdot ?$
(4) $9ay - 9by + 27cy = (9y) \cdot ?$

4:10

4:10 Write the values of the products and quotients. Check the quotients by multiplying.

Mentally
40 × 20
30 × 11
12 × 60
5 × 19
480 ÷ 8

With pencil and paper
6,132 × 6
48 × 39
7) 8,722

4:5

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} =$ ___ (a)
$6 \times \frac{4}{7} =$ ___ (b)
$86 \times \frac{1}{86} =$ ___ (c)
$6 \times \frac{8}{2} =$ ___ (d)
$9 \times \frac{1}{9} =$ ___ (e)
$9 \times \frac{2}{9} =$ ___ (f)

(1g) Which answer is twice as much as the answer for (e)?
(1h) Which answer is six times as much as the answer for (a)?
(1i) Which two answers are equal?
(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

3:10

3:10 Alice forgot what 7 × 8 equals. Alice knows that 5 × 8 = 40 and 2 × 8 = 16.
(1) Write a sentence to tell Alice how she can find the value of 7 × 8 by using the two facts she knows.
(2) Draw a diagram that could help Alice understand why your method works.
(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

Another grade 5 task that prominently features the distributive property is **5:14 Brandon's Equation**.

In later grades, the distributive property will become the central principle in rewriting expressions; see tasks **6:11 Area Expressions** and **7:3 Writing Sums as Products**.

In earlier grades, tasks **4:10 Calculating Products and Quotients, 4:5 Fraction Products and Properties,** and **3:10 Alice's Multiplication Fact** demonstrate how the distributive property is used to do mental math, organize calculations, and extend operations from whole numbers to fractions.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:1? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 5:1? In what specific ways do they differ from 5:1?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:1 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

Central math concepts

In earlier grades, students might have considered a situation with some resemblance to task 5:2, but with a discrete rather than continuous quantity—such as,

> Neighbors donated 40 picture frames to the 12 residents of a nursing home. That would provide ______ picture frames for each resident and ______ picture frames for the common room.

This situation could be described with an equation model such as

$$40 = 3 \times 12 + 4,$$

where the units of the quotient are "frames per resident" and the units of the remainder are "picture frames." Alternatively, the situation could be described with an equation model such as

$$(40 - 4) \div 12 = 3,$$

where the units of the quotient are again "frames per resident" and the units of the part subtracted are "picture frames." Notice that the only numbers that appear in these two equation models are whole numbers, which is sensible because fractional parts of picture frames aren't part of the situation. Notice also that in the second equation model that uses division, we could check the quotient by multiplying: $3 \times 12 = 40 - 4$ is a true equation.

The equation $40 = 3 \times 12 + 4$ could also describe a continuous quantity like water. In that case, the equation $40 = 3 \times 12 + 4$ says that with a supply of 40 gallons of water, each of 12 residents could be given 3 gallons of water, and there would be 4 gallons left over. However, unlike picture frames, when the quantity in a situation is continuous, we can share the remainder also. That is, the equation

$$40 = 3 \times 12 + 4$$

implies

$$40 \div 12 = (3 \times 12) \div 12 + 4 \div 12,$$

or

$$40 \div 12 = 3 + \tfrac{1}{3}.$$

In this last equation, the units of 3, $\frac{1}{3}$, and $3 + \frac{1}{3}$ are all the same: "gallons per resident." The remainder 4 had units of "gallons," whereas after division by 12, the resulting fractional part $\frac{1}{3}$ has units "gallons per resident."

Task 5:2 thus involves the extension of the number system from whole numbers to fractions. This extension is necessary if we are to share continuous quantities like water equally, and it also solves a conceptual problem that arises as early as grade 3. Consider the dilemma of a third-grade student who solves problems like $72 \div 9 = 8$ but who may wonder what to make of a problem like $72 \div 10 = ?$. The unknown factor can't be

5:2 After a hurricane, the 12 residents of a nursing home didn't have any clean water to drink. Their neighbors donated 40 gallons of bottled water, which would provide ____ gallons for each resident.

Answer

$3\frac{1}{3}$ or the equivalent (such as $\frac{10}{3}$).

Click here for a student-facing version of the task.

Refer to the Standards

5.NF.B.3; MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

In a challenging life situation like the one described in the task, it would make sense not to waste the $\frac{1}{3}$ fractional part of a gallon per person.

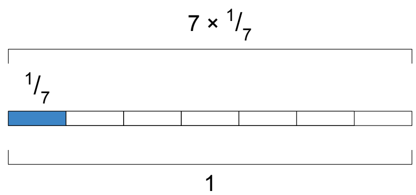
Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:2? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:2? In what specific ways do they differ from 5:2?

7 (because 10 × 7 = 70, which is too small); and the unknown factor can't be 8 (because 10 × 8 = 80, which is too large). So maybe the unknown factor is between 7 and 8? Closer to 7, because 70 is closer to 72 than 80 is? It isn't until the upper-elementary grades that students fully integrate positive fractions into their expanding system of numbers and operations.

Fractions allow division of any whole number C by any nonzero whole number A. Because $C = A \times \frac{C}{A}$, the quotient $C \div A$ is the number $\frac{C}{A}$, because $C \div A$ is the unknown factor in $C = A \times ?$. It may seem obvious that $\frac{C}{A} = C \div A$ if $\frac{C}{A}$ is already thought of as an instruction to calculate $C \div A$, but the statement $\frac{C}{A} = C \div A$ is more meaningful if the fraction $\frac{C}{A}$ is thought of as a number before it is recognized as a quotient.

An important special case of the principle $\frac{C}{A} = C \div A$ is dividing 1 by a nonzero whole number. For example, $1 \div 7 = \frac{1}{7}$ (see figure). This result can be extended to a quotient like $5 \div 7$ by means of diagrams and/or by thinking that, since 5 is five times as much as 1, then $5 \div 7$ is five times as much as $1 \div 7$, or 5 parts of size $\frac{1}{7}$, which is to say $\frac{5}{7}$.



Extending the number system from the whole numbers to the fractions allows answers to questions like the one our hypothetical third-grade student was wondering about: the answer to $72 \div 10 = ?$ is the number $\frac{72}{10}$. The relationship between multiplication and division—that $C \div A$ is the unknown factor in $A \times ? = C$—is a trustworthy guide to mathematical reasoning, used frequently beginning in grade 3 and continuing at least until grade 7. The relationship recurs in high school as well, with the study of rational expressions, rational functions, and complex numbers.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: multiplying unit fractions; making sense of products of unit fractions; applying ideas of times-as-much in context; and basing multiplicative reasoning on math diagrams.

Extending the task

How might students drive the conversation further?

- Students could check the quotient $3\frac{1}{3}$ or $\frac{10}{3}$ by multiplying.
- Students could work the problem by imagining that the water was donated in bottles that each hold $\frac{1}{3}$ gallon. How many of those bottles were donated? How many bottles does each of the 12 residents get?

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:2 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

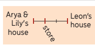
* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

- Students could describe how the $\frac{1}{3}$ gallon fractional part could be physically provided to the residents (at least approximately), supposing that all the donations were in the form of gallon bottles. How do the quantities in the distribution process correspond to the numbers in the equation 40 = 3 × 12 + 4?

Related Math Milestones tasks

5:6

5:6 **(1)** Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?



5:13

5:13 In a snack shop there is a frozen yogurt machine. When there is 3 *l* of frozen yogurt in the machine, the machine is $\frac{1}{3}$ full. How much frozen yogurt is in the machine when it is $\frac{1}{4}$ full?

5:5

5:5 Write the requested values.

4087 × 53 = ?	$\frac{1}{10} \div 10 = ?$	0.4 × 0.9 = ?
246 × 914 = ?	$\frac{7}{8} \times \frac{5}{3} = ?$	0.75 ÷ 0.01 = ?
9744 ÷ 12 = ?	8 × ? = 73	0.63 ÷ 0.3 = ?
1461 ÷ 6 = ?	$3 \div \frac{1}{8} = ?$	0.86 + 0.4 = ?
4 − (8 − 4) = ?	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	0.72 − 0.17 = ?
	$\frac{1}{3} \div (6 \times 5) = ?$	0.02 + 0.2 = ?
		0.8 − 0.55 = ?
		637 − 1.31 = ?

Task **5:6 Corner Store** involves fraction products in context. Task **5:13 Frozen Yogurt Machine** is a multi-step word problem that involves multiplicative thinking with fractions. Task **5:5 Calculating** includes the procedural problems 1461 ÷ 6 and 8 × ? = 73 that involve quotients of whole numbers.

6:1

6:1 $\frac{2}{3}$ of a charging cord is $\frac{1}{2}$ meter long. How long is the charging cord? (Answer in meters.)

6:9

6:9 How much of a $\frac{3}{4}$-ton truckload is $\frac{2}{3}$ ton of gravel?

6:12

6:12 **(1)** What is the area of the triangle in the coordinate plane with vertices (1, 2), (−5, 2), and (−8, 9)? **(2)** How does the area change if we change the third vertex to (−3, 9)?

In later grades, tasks **6:1 Charging Cord** and **6:9 Truckload of Gravel** are word problems that use multiplication and division operations on fractions, completing the extension of arithmetic from whole numbers to fractions. Task **6:12 Is There a Solution? (Multiplication)** looks at this extension from an algebra perspective.

4:5

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} = \underline{\quad}$ (a)

$6 \times \frac{4}{7} = \underline{\quad}$ (b)

$86 \times \frac{1}{86} = \underline{\quad}$ (c)

$6 \times \frac{8}{2} = \underline{\quad}$ (d)

$9 \times \frac{1}{9} = \underline{\quad}$ (e)

$9 \times \frac{2}{9} = \underline{\quad}$ (f)

(1g) Which answer is twice as much as the answer for (e)?

(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

3:6

3:6 Using what you know about fractions, decide which is greater, $\frac{1}{73}$ or $\frac{1}{41}$. Tell how you decided.

In earlier grades, task **4:5 Fraction Products and Properties** involves the initial stages of extending multiplication to fractions. The building blocks of fractions are the topic in task **3:6 Unit Fraction Ideas**.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
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Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:3 Neighborhood Garden

Teacher Notes

Central math concepts

In grades K–2, students work with length units and concepts of length measurement (CCSS 1.MD.A.2). At first in kindergarten, students make non-numerical comparisons of length and other measurable quantities (CCSS K.MD.A). Then in grade 1, using objects as length units, students learn that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. In grade 2, students extend these ideas to abstract length units and relate length measurement to addition, subtraction, and the number line (CCSS 2.MD.A).

Beginning in grade 3, students learn to recognize area as a measurable attribute of plane figures and to understand concepts of area measurement (CCSS 3.MD.C.5):

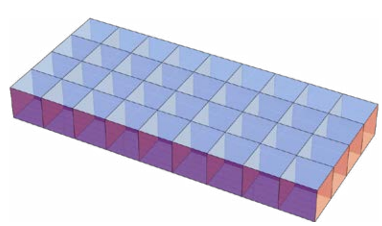
- A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
- A plane figure that can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units.

Observe the close parallel to volume in grade 5 (CCSS 5.MD.C.3):

- A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.
- A solid figure that can be packed without gaps or overlaps using *n* unit cubes is said to have a volume of *n* cubic units.

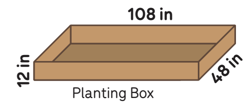
Additional parallels can also be seen for angle measure in grade 4 (CCSS 4.MD.C.5; see the Teacher Notes for task **4:8 Shapes with Given Positions**). Measurement, and especially the idea of a unit, is a coherent and unifying theme in school mathematics.†

WIth the length unit as 1 inch, the planting box in task 5:3 could be packed with 12 layers of unit cubes, each layer consisting of an array of unit cubes viewed as 48 rows and 108 unit cubes in each row (see figure; click here for a larger image). Each layer has 48 × 108 unit cubes, and the 12 layers have 48 × 108 × 12 unit cubes total. Thus the volume is 48 × 108 × 12 cubic inches.



Had we chosen the length unit to be 1 foot, then we could pack the planting box with a single array of unit cubes, this array viewed as 4 rows and 9 unit cubes in each row (see figure; click here for a larger image). Thus, the volume is 4 × 9 cubic feet. To emphasize the correspondence with the three-factor volume formula for a rectangular prism,

5:3 A neighborhood garden will have 6 wooden planting boxes. Every box will have the same shape (see diagram). Soil can be bought by the truckload; a truckload is 54 ft^3 of soil. How many truckloads of soil will fill all of the boxes?



Answer

4 truckloads.

Click here for a student-facing version of the task.

Refer to the Standards

5.MD.A, B; MP.1, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

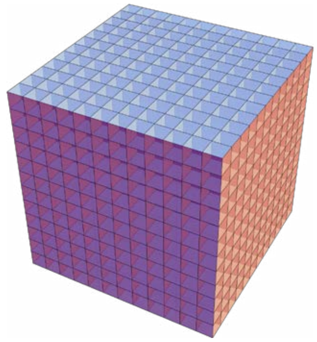
Aspect(s) of rigor:

Application

Additional notes on the design of the task

The numbers in the problem are chosen to have many divisibilities so that, for example, students could solve the problem by analyzing it in the form 6 × 9 × ? = 6 × 4 × 9.

$V = \ell wh$, we could use the multiplicative identity property of 1 to write the product 4×9 in the equivalent form $4 \times 9 \times 1$.



Each cubic foot contains $12 \times 12 \times 12$ cubic inches (see figure; click here for a larger image). Seeing each cubic foot as $12 \times 12 \times 12$ cubic inches can help make sense of the fact that $48 \times 108 \times 12$ cubic inches is the same quantity of volume as $4 \times 9 \times 1$ cubic feet:

$$\begin{aligned} 48 \times 108 \times 12 &= (12 \times 4) \times (12 \times 9) \times (12 \times 1) \\ &= (12 \times 12 \times 12) \times (4 \times 9 \times 1). \end{aligned}$$

Here the associative and commutative properties of multiplication have been used.

Because the truckloads of soil are given in cubic feet, whereas the planting box dimensions are given in inches, students have to choose which unit to use. Working in units of feet and cubic feet makes the problem much easier to solve, and this is an illustration of how important it is for students to learn they must make choices when solving problems. Students can learn how to choose convenient units in applied problems by experiencing and getting feedback on choosing.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: the equal-groups idea of multiplication; remembering products of single-digit numbers; and finding simple quotients.

Extending the task

How might students drive the conversation further?

- Students might know that, or think about the fact that, soil is often sold by the cubic yard. How many cubic feet is 1 cubic yard? How many cubic yards is one of the truckloads in task 5:3?
- Students might consider making truckload-units central in the problem. How many boxes does a truck hold? ($54 = 36 \times n \Rightarrow n = 54 \div 36 = \frac{54}{36} = \frac{3}{2}$.) Therefore, how many truckloads are needed for 6 boxes? 4 truckloads, because $4 \times \frac{3}{2} = 6$.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:3? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:3? In what specific ways do they differ from 5:3?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:3 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† See Zimba (2013), "Units, a Unifying Theme in Measurement, Fractions, and Base Ten."

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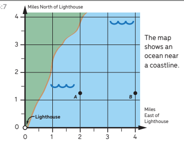
Related Math Milestones tasks

5:7

5:7

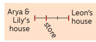


The map shows an ocean near a coastline.

Shipwrecks are at locations $A\ (2, 1\frac{1}{4})$ and $B\ (4, 1\frac{1}{4})$. Shipwrecks are also at locations $C\ (4, 3\frac{1}{2})$ and $D\ (2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

5:6

5:6

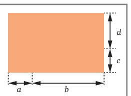


(1) Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?

Task **5:7 Shipwrecks** involves geometric measure in a case with fractional dimensions, and task **5:6 Corner Store** involves an equality between two products that depends on the commutative and associative properties of multiplication.

6:11

6:11 The diagram shows a rectangle. The variables a, b, c, and d are lengths in meters. **(1)** Using the variables, write three different expressions for the area of the rectangle. **(2)** Choose two of your expressions and show that they are equivalent by applying properties of operations. **(3)** State the property or properties you used.



6:12

6:12 **(1)** What is the area of the triangle in the coordinate plane with vertices (1, 2), (−5, 2), and (−8, 9)? **(2)** How does the area change if we change the third vertex to (−3, 9)?

7:13

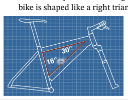
7:13 A 15.1-in long wire is bent into the shape of a circle with 2.9 in left over. To the nearest 0.1 in, what is the diameter of the circle?

7:10

7:10 In ΔABC, side AB is 4 units long, side BC is 3 units long, and angle A measures 30°. Sketch two ways ΔABC might look.

8:3

8:3 On this blueprint for building a bike, part of the bike is shaped like a right triangle. The longest side length is illegible because water spilled on the blueprint. Calculate that side length.



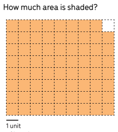
8:12

8:12 Design a fish tank that fits into the corner of a room. Use a quarter of a cylinder as a model for the tank. To share your design, make a diagram showing the tank measurements. Also, calculate the weight of the water when your tank is filled (1 m^3 of water weighs about 1,000 kg). Write your calculation steps so that a classmate could understand how you did it.

In later grades, task **6:11 Area Expressions** involves a geometric measure in a case where the lengths are variables rather than numbers; task **6:12 Coordinate Triangle** involves area measure for a triangle; task **7:13 Wire Circle** involves a geometric measure of a curvilinear object; task **7:10 Triangle Conditions** involves length and angle measures in a triangle; task **8:3 Bicycle Blueprint** involves the Pythagorean theorem, and task **8:12 Fish Tank Design** involves volume measurement for a quarter-cylinder.

3:3

3:3 **(1)** How much area is shaded?



(2) Using a ruler, draw a rectangle with area 28 square centimeters. Write the length and width of your rectangle.

Length: ________ Width: ________

4:13

4:13 **(1)** A red rectangle has length $L = 12$ in and width $W = 6$ in. Use the formula $A = L \times W$ to find the area of the red rectangle.

(2) A blue rectangle has length 1 ft and width $\frac{1}{2}$ ft. Draw a picture to show that two copies of the blue rectangle make one square foot. Based on your picture, what is the area of the blue rectangle?

(3) Do the red rectangle and the blue rectangle have equal areas? Tell how you decided.

In earlier grades, tasks **3:3 Length and Area Quantities** and **4:13 Area Units** involve concepts of area measurement.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:4 Place Value to Thousandths

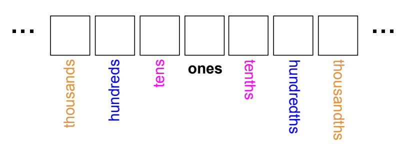
Teacher Notes

Central math concepts

In the place value system, a digit in one place of a multi-digit number represents 10 times as much as it represents in the place to its right. Equivalently, a digit in one place of a multi-digit number represents $\frac{1}{10}$ of what it represents in the place to its left. This description applies to both whole numbers and decimals.

1 is the unit from which the other units are created; 1 corresponds to what you are counting: what counts as one…the unit…in any given context. So in a string of digits like 30485, it is essential to know which place is the ones place. [In the Middle Ages](), the ones place was often indicated by an overline. With this convention, $30\overline{4}85$ indicates the number 304.85. The overline convention is mathematically interesting because it highlights the symmetry of the place value system about the ones place (see the diagram).

Symmetry of the Place Value System About the Ones Place



Because of their compatible relative sizes, base-ten units can be bundled or unbundled into other base-ten units. Bundling and unbundling are central ideas in developing algorithms for calculation in base-ten. As noted in the *Progression* document,[†] the compatibility of place value units also leads to multiple ways to refer to a decimal number, as when we refer to 0.15 as "15 hundredths" or equivalently as "1 tenth and 5 hundredths."

A decimal calculation can be performed (or justified) by rewriting the problem in fraction notation, performing the fraction calculation, and writing the result as a decimal. For example, $0.4 \times 0.9 = \frac{4}{10} \times \frac{9}{10} = \frac{4 \times 9}{10 \times 10} = \frac{36}{100} = 0.36$. However, the enormous practical benefit of the decimal system is that because the system is identical at each place, algorithms developed for whole-number computations can be applied or easily adapted to handle calculations with decimals.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: reasoning about unit fractions, fraction equivalence, and products of unit fractions; and writing fractions as decimals and vice versa.

Extending the task

How might students drive the conversation further?

- Students could create their own versions of task 5:4 by keeping the digits the same but mixing up the place value units. For example, part (2) (a) could be mixed up to read "7 tenths + 5 thousandths = _____ ."

5:4

(1) Circle T for true or F for false.

(a) 9 thousandths + 5 hundredths > 3 hundredths + 2 tenths   T F

(b) 92 hundredths + 4 thousandths > 0.924   T F

(c) 0.456 < 0.5   T F

(2) Write each number in the requested form.

(a) 7 thousands + 5 tenths = _____ (decimal)

(b) 0.1 tenths = _____ (decimal)

(c) $\frac{2}{100} + \frac{5}{1000}$ = _____ (decimal) = _____ (fraction in lowest terms)

Answer

(1) (a) False. **(b)** False. **(c)** True.

(2) (a) 0.507. **(b)** 0.01. **(c)** $0.025 = \frac{1}{40}$.

Click here for a student-facing version of the task.

Refer to the Standards

5.NBT.A; MP.1, MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- Task 5:4 is designed to target conceptual understanding, even though it only asks for brief answers rather than extended writing or other language demands. Teachers can also question students about the thinking that led to their answers, individually or in a group setting (and students can question each other).

Students could trade their problems with a partner and check each other's answers.

- Students might know about countries where decimal separators other than the period are used. Students could show how numbers are written in other systems they are familiar with.
- Students could show what 7 tenths + 5 hundredths looks like on the number line.

Related Math Milestones tasks

5:5

5:5 Write the requested values.

$4087 \times 53 = ?$
$246 \times 914 = ?$
$9744 \div 12 = ?$
$1461 \div 6 = ?$
$4 - (8 - 4) = ?$

$\frac{1}{10} \div 10 = ?$
$\frac{7}{8} \times \frac{5}{3} = ?$
$8 \times ? = 73$
$3 \div \frac{1}{8} = ?$
$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$
$\frac{1}{3} \div (6 \times 5) = ?$

$0.4 \times 0.9 = ?$
$0.75 \div 0.01 = ?$
$0.63 \div 0.3 = ?$
$0.86 + 0.4 = ?$
$0.72 - 0.17 = ?$
$0.02 + 0.2 = ?$
$0.8 - 0.55 = ?$
$637 - 1.31 = ?$

5:10

5:10 **(1)** Solve: $\frac{1}{3} = 0.1 + ?$

(2) Is there a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$? If you think so, find such a number. If you think there is no such number, explain why.

(3) Show one of the above problems and its solution on a number line.

6:14

6:14 *Pencil and paper* **(1)** $81.53 \div 3.1 = ?$ **(2)** $\frac{7}{8} \div \frac{2}{3} = ?$ **(3)** Check both of your answers by multiplying.

4:7

4:7 Write the values of the expressions. Read each completed equation aloud.

3 fifths + 2 fifths = ____

$\frac{1}{10} + \frac{3}{100} =$ ____ (fraction)
$=$ ____ (decimal)

$\frac{6}{25} + \frac{6}{25} =$ ____

$\frac{1}{8} + \frac{5}{8} - \frac{3}{8} =$ ____

2:2

2:2 **(1) True or false?**

(a) 2 hundreds + 3 ones > 5 tens + 9 ones
(b) 9 tens + 2 hundreds + 4 ones < 924
(c) 456 < 5 hundreds

(2) Write the number that makes each statement true.

(a) 7 ones + 5 hundreds = ____
(b) 14 tens = ____
(c) 90 + 300 + 4 = ____

Task **5:5 Calculating** involves several procedural problems that rest on understanding of decimal place value. Task **5:10 Number System, Number Line** aims at synthesis of fractions and decimals into a unified conception of number supported by the number line.

In later grades, task **6:14 Dividing Decimals and Fractions** marks the culmination of decimal procedures with a decimal division calculation.

In earlier grades, task **4:7 Fraction Sums and Differences** involves unit thinking about fractions including fractions with denominators of 10 and 100. Task **2:2 Place Value to Hundreds** is an analogue of task 5:4 that involves base ten units of hundreds, tens, and ones.

Additional notes on the design of the task (continued)

- Place value units of tenths, hundredths, and thousandths appear in several different orders so that relating the named quantities to base-ten numerals in positional notation is part of the task.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:4? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:4? In what specific ways do they differ from 5:4?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:4 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2015, March 6). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5, Number and Operations in Base Ten*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
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- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

Central math concepts

Task 5:5 focuses on procedures. For each individual computation included in task 5:5, students have a choice between using an algorithm or a strategy.

Algorithms are usefully distinguished from strategies (CCSS, Glossary; see figure). Strategies are "purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another." Mental calculation often uses strategies. For example, we could calculate 0.75 ÷ 0.01 mentally by using understanding of place value to replace the given quotient by the equivalent 75 ÷ 1. Or we could calculate $3 \div \frac{1}{8}$ by thinking that since there are 8 eighths in every 1, there must be 24 eighths in 3. Flexible procedural fluency based on place value, properties of operations, relationships between operations, and ideas of equivalence is valuable both in practical terms and also as a way of doing arithmetic that prefigures algebra. In algebra, strategies are the most common way of working.

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Algorithms are inflexible by definition. One step follows another in the prescribed order. Some algorithms are also much simpler to execute than others. The standard multi-digit addition algorithm is less complex than the standard multi-digit subtraction algorithm. The standard algorithm for dividing fractions is less complex than the standard algorithm for dividing multi-digit numbers.

Sometimes even an efficient general-purpose algorithm wouldn't be an efficient approach to a particular instance of a calculation, as in a subtraction problem like 4,003 - 8. Instead of applying the algorithm $\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$ to the problem $\frac{1}{2} + \frac{1}{3} - \frac{1}{5}$ in task 5:5, one can use the principle of fraction equivalence to replace the given problem with a problem in which the denominators are equal: $\frac{15}{30} + \frac{10}{30} - \frac{6}{30}$.

On the other hand, when faced with a calculation, there may be times when we don't find ourselves readily inventing a flexible mental procedure on the spot, so it's valuable to know and be proficient with an algorithm.

An important value in mathematics education is that of being able to solve problems in multiple ways. This brings the pleasures of seeing how a coherent subject holds together, and it allows students to check answers, unify their understanding of concepts, and learn from different ways of thinking that emerge in the classroom community. A parallel but also important outcome of mathematics education is for students to be supported in gaining procedural fluency with algorithms for the actually

5:5 Write the requested values.

4087 × 53 = ?	$\frac{1}{10} \div 10 = ?$	0.4 × 0.9 = ?
246 × 914 = ?	$\frac{7}{8} \times \frac{5}{3} = ?$	0.75 ÷ 0.01 = ?
9744 ÷ 12 = ?	8 × ? = 73	0.63 ÷ 0.3 = ?
1461 ÷ 6 = ?	$3 \div \frac{1}{8} = ?$	0.86 + 0.4 = ?
4 – (8 – 4) = ?	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	0.72 – 0.17 = ?
	$\frac{1}{3} \div (6 \times 5) = ?$	0.02 + 0.2 = ?
		0.8 – 0.55 = ?
		637 – 1.31 = ?

Answer

4087 × 53 = **216,611**
246 × 914 = **224,844**
9744 ÷ 12 = **812**
1461 ÷ 6 = $\mathbf{243\frac{1}{2}}$ **or 243.5**
4 - (8 - 4) = **0**

$\frac{1}{10} \div 10 = \mathbf{\frac{1}{100}}$
$\frac{7}{8} \times \frac{5}{3} = \mathbf{\frac{35}{24}}$
8 × ? = 73: ? = $\mathbf{\frac{73}{8}}$ or $\mathbf{9\frac{1}{8}}$
$3 \div \frac{1}{8} = \mathbf{24}$
$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = \mathbf{\frac{19}{30}}$
$\frac{1}{3} \div (6 \times 5) = \mathbf{\frac{1}{90}}$

0.4 × 0.9 = **0.36**
0.75 ÷ 0.01 = **75**
0.63 ÷ 0.3 = **2.1**
0.86 + 0.4 = **1.26**
0.72 - 0.17 = **0.55**
0.02 + 0.2 = **0.22**
0.8 - 0.55 = **0.25**
637 - 1.31 = **635.69**

Click here for a student-facing version of the task.

Refer to the Standards

5.NBT.B, 5.NF.A, B; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

quite small set of recurrent problem types for which an algorithm exists. This small set can be found in the CCSS-M by searching for "algorithm."

Many people are not perfectionists in execution; this doesn't mean they should be identified as "bad at math." Anxiety and low confidence can also contribute to errors. Thus, errors in calculations are sometimes the result of careless mistakes rather than gaps in understanding. To reduce the rate of errors, students can learn general and specific debugging skills. Debugging involves reflecting on the execution of the specific mistake and formulating a behavior that is a correct execution for similar situations. This can be much more efficient than reteaching topics.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: place value, number sense of decimals, and decimal notation; number sense of fractions; multiplication facts and related quotients; and the relationship between multiplication and division.

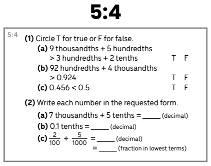
Extending the task

How might students drive the conversation further?

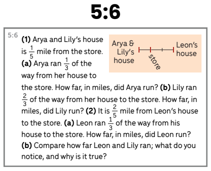
- Checking differences by adding, and checking quotients by multiplying, can offer additional procedural practice, reinforce the relationship between addition and subtraction ($C - A$ is the unknown factor in $A + \square = C$), and reinforce the relationship between multiplication and division ($C \div A$ is the unknown factor in $A \times \square = C$).
- Students could make sense of their answers another way by making estimates of the values; for example, 246 × 914 should be somewhat less than 246 × 1,000 = 246,000; 0.63 ÷ 0.3 should be slightly greater than 0.6 ÷ 0.3 = 2.

Related Math Milestones tasks

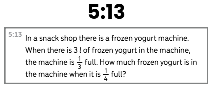
5:4


5:4 (1) Circle T for true or F for false.
(a) 9 thousandths + 5 hundredths > 3 hundredths + 2 tenths T F
(b) 92 hundredths + 4 thousandths > 0.924 T F
(c) 0.456 < 0.5 T F
(2) Write each number in the requested form.
(a) 7 thousandths + 5 tenths = _____ (decimal)
(b) 0.1 tenths = _____ (decimal)
(c) $\frac{2}{100} + \frac{5}{1000}$ = _____ (decimal) = _____ (fraction in lowest terms)

5:6

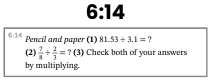

5:6 (1) Arya and Lily's house is $\frac{1}{5}$ mile from the store. (a) Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? (b) Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? (2) It is $\frac{2}{5}$ mile from Leon's house to the store. (a) Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? (b) Compare how far Leon and Lily ran; what do you notice, and why is it true?

5:13


5:13 In a snack shop there is a frozen yogurt machine. When there is 3 *l* of frozen yogurt in the machine, the machine is $\frac{1}{3}$ full. How much frozen yogurt is in the machine when it is $\frac{1}{4}$ full?

Task **5:4 Place Value to Thousandths** involves decimal place value concepts, while tasks **5:6 Corner Store** and **5:13 Frozen Yogurt Machine** involve fraction multiplication and division in context.

6:14


6:14 *Pencil and paper* (1) 81.53 ÷ 3.1 = ? (2) $\frac{7}{8} \div \frac{2}{3} = ?$ (3) Check both of your answers by multiplying.

In later grades, task **6:14 Dividing Decimals and Fractions** marks the culmination of algorithmic procedures with general fraction and decimal division.

Aspect(s) of rigor:

Procedural skill and fluency

Additional notes on the design of the task

- Some of the problems where buggy execution may occur could include 4087 × 53 (because of the 0 digit), 4 - (8 - 4) (because of the subtraction sign in front of parentheses), and the decimal problems in which the two operands have differing numbers of digits.
- The task does not require students to show their work. For some of the calculations, students might simply write down the answer by inspection. For other calculations, students might write out steps along the way to finding the result.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:5? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 5:5? In what specific ways do they differ from 5:5?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:5 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

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4:10

4:10 Write the values of the products and quotients.
Check the quotients by multiplying.

Mentally

40×20
30×11
12×60
5×19
$480 \div 8$

With pencil and paper

$$\begin{array}{r} 6{,}132 \\ \times \quad 6 \\ \hline \end{array} \qquad \begin{array}{r} 48 \\ \times\ 39 \\ \hline \end{array} \qquad 7\overline{)8{,}722}$$

4:14

4:14 $540{,}909 + 87{,}808 - 5{,}864 + 2{,}556 = ?$

In earlier grades, task **4:10 Calculating Products and Quotients** includes a range of procedural tasks amenable to grade-level techniques. Task **4:14 Fluency with Multi-Digit Sums and Differences** involves the culmination of multi-digit addition and subtraction procedures.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
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Language

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Identity, Agency, and Belonging

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- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:6 Corner Store

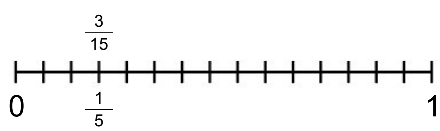
Teacher Notes

Central math concepts

Part (1)(a) of task 5:6 involves multiplying the unit fractions $\frac{1}{3}$ and $\frac{1}{5}$ in context. Making sense of a product of unit fractions is a special moment in a student's mathematical education, a moment of entry into a newly sophisticated world of interrelated numbers and operations. A product like $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$ doesn't sit comfortably alongside third-grade understandings of products—such as, say, interpreting 6 × 5 as the total number of marbles in 6 bags if each bag contains 5 marbles. For one thing, the product $\frac{1}{3} \times \frac{1}{5}$ is less than either of its factors, something that never happens with marbles. And yet multiplication is a single operation with a meaning that transcends the size or format of the numbers being multiplied. Synthesizing students' intuitions about multiplication is part of the important work in grade 5.

Fortunately, conceptual bridges are available to help students make the journey from multiplication ideas that only work for whole numbers to multiplication ideas that work for all numbers. One such bridge is using representations like area models and number lines, which are well suited to both whole-number products and products involving fractions. Fraction equivalence is a powerful tool for reasoning about fraction operations. Another bridge is the multiplicative concept of "times as much," which could be seen as intermediate between the grade 3 discrete equal-groups concept of multiplication and the grade 5 scaling concept of multiplication. Times-as-much thinking is well suited to contexts involving continuous measurement quantities like length, time, and mass or weight, and quantities derived from these by multiplication and division. These are the kinds of quantities for which fractional parts most naturally arise.

The diagram shows an equivalent-fractions approach to making sense of the product $\frac{1}{3} \times \frac{1}{5}$, using a length model to organize the thinking.

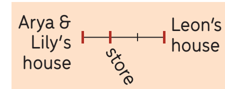


One-third of $^1/_5$ means 1 part of a partition of $^1/_5$ into 3 equal parts. To facilitate partitioning $^1/_5$ into 3 equal parts, one can use the fraction equivalence principle to rewrite $^1/_5$ as $^3/_{15}$. Because $^1/_5$ is now expressed as 3 of something (namely 3 fifteenths), it is more apparent that in a partition of $^1/_5$ into 3 parts, 1 of the parts will have size $^1/_{15}$. Thus, $^1/_3 \times {}^1/_5 = {}^1/_{15}$.

Eventually the kinds of reasoning processes reflected in the diagram cease to be the focus of a student's work with fractions, and calculating products of fractions becomes a routine exercise in procedural fluency. However, extending multiplication and division from whole numbers to fractions is a momentous mathematical journey that begins not with procedure but with reasoning, sense-making, and quantitative thinking.

Part (2) of task 5:6 involves the equality of the products $\frac{2}{3} \times \frac{1}{5}$ and $\frac{1}{3} \times \frac{2}{5}$. If we think of the first product as $2 \times (\frac{1}{3} \times \frac{1}{5})$ and the second product as

5:6 **(1)** Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?



Answer

(1) (a) $\frac{1}{15}$ mi. **(b)** $\frac{2}{15}$ mi.

(2) (a) $\frac{2}{15}$ mi. **(b)** Leon and Lily ran the same distance measured in miles. Explanations for this will vary. One kind of explanation could involve recognizing that doubling and then halving a given length results in the given length. Another kind of explanation could involve the algorithm for multiplying fractions. Answers may include such explanatory techniques as showing a math diagram (for example, an area model, tape diagram, or number line), writing expressions and equations, or using analogies (for example, "If I my water bottle is $\frac{1}{3}$ full and yours is $\frac{2}{3}$ full, but my water bottle is twice as big as yours, then we have the same amount of water").

Click here for a student-facing version of the task.

Refer to the Standards

5.NF.B.4a, 5.NF.B.6; MP.2, MP.3, MP.4, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

$\frac{1}{3} \times (2 \times \frac{1}{5})$, then the equality $\frac{1}{3} \times (2 \times \frac{1}{5}) = 2 \times (\frac{1}{3} \times \frac{1}{5})$ reflects the fact that one-third of twice a given quantity is twice as much as one-third of the quantity. (There would be no preference between two-thirds of one sum of money and one-third of another sum of money, provided the second sum of money were twice as large.) This could again be justified by a diagram and/or by using equivalent fractions, and viewed as an example of the associative and commutative properties of multiplication:

$$\frac{2}{3} \times \frac{1}{5} = (2 \times \frac{1}{3}) \times \frac{1}{5} = (\frac{1}{3} \times 2) \times \frac{1}{5} = \frac{1}{3} \times (2 \times \frac{1}{5}) = \frac{1}{3} \times \frac{2}{5}.$$

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: multiplying unit fractions; making sense of products of unit fractions; applying ideas of times-as-much in context; and basing multiplicative reasoning on math diagrams.

Extending the task

How might students drive the conversation further?

- Students who gave different accounts in part (2)(b), or used different representations, could look for correspondences between them.
- Students could ask and answer additional questions about the situation, such as, "If Leon walked halfway from his house to Arya's house, how far from the store would Leon be?"

Related Math Milestones tasks

5:13

5:13 In a snack shop there is a frozen yogurt machine. When there is 3 *l* of frozen yogurt in the machine, the machine is $\frac{1}{3}$ full. How much frozen yogurt is in the machine when it is $\frac{1}{4}$ full?

5:2

5:2 After a hurricane, the 12 residents of a nursing home didn't have any clean water to drink. Their neighbors donated 40 gallons of bottled water, which would provide ____ gallons for each resident.

5:11

5:11 Juliet said, "I'm thinking of a rectangle. Its area is 1 square unit. Its perimeter is more than 1 million units."
(1) Is Juliet thinking of something possible or impossible? Use math to decide for sure.
(2) Explain your reasoning to your classmates. Revise your explanation based on suggestions from your classmates.

Task **5:13 Frozen Yogurt Machine** is a multi-step word problem that involves multiplicative thinking with fractions. Task **5:2 Water Relief** involves interpreting the quotient of two whole numbers as a fraction. Task **5:11 Juliet's Rectangle** could promote thinking about the extension of multiplication from whole numbers to fractions.

6:1

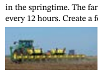
6:1 $\frac{2}{3}$ of a charging cord is $\frac{1}{2}$ meter long. How long is the charging cord? (Answer in meters.)

6:9

6:9 How much of a $\frac{3}{4}$-ton truckload is $\frac{2}{3}$ ton of gravel?

6:6

6:6 A farmer uses a tractor to plant corn quickly in the springtime. The farmer plants 216 acres every 12 hours. Create a formula for the number of acres the farmer plants in n hours.



6:13

6:13 *Pencils down* Think about the equation $241p = \frac{3}{4}$. Is there a whole number that solves it? Is there a non-whole number that solves it? Convince a classmate that your answers are right.

In later grades, tasks **6:1 Charging Cord** and **6:9 Truckload of Gravel** are word problems that use multiplication and division operations on fractions, completing the extension of arithmetic from whole numbers to fractions. Task **6:6 Planting Corn** is about proportionality (can involve application of times-as-much thinking). In task **6:13 Is There a Solution? (Multiplication)**, the equation can be made sense of as a question about multiplicative scaling.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

In part (2)(b), the phrasing "why is it true?" is intended to invite sense-making about the observation that Leon and Lily ran the same distance measured in miles. The intent is to push beyond an algorithmic demonstration that $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15} = \frac{1}{3} \times \frac{2}{5}$ and think about the sizes of the quantities and the meaning of the operation of multiplication.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:6? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:6? In what specific ways do they differ from 5:6?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:6 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

4:5

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} =$ ____ (a)

$6 \times \frac{4}{7} =$ ____ (b)

$86 \times \frac{1}{86} =$ ____ (c)

$6 \times \frac{8}{2} =$ ____ (d)

$9 \times \frac{1}{9} =$ ____ (e)

$9 \times \frac{2}{9} =$ ____ (f)

(1g) Which answer is twice as much as the answer for (e)?

(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

4:1

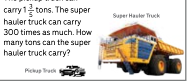
4:1 A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ____________

Answer: ____________

4:12

4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?



3:6

3:6 Using what you know about fractions, decide which is greater, $\frac{1}{73}$ or $\frac{1}{41}$. Tell how you decided.

In earlier grades, task **4:5 Fraction Products and Properties** can involve times-as-much thinking and sense-making about fraction products. Tasks **4:1 A Tablespoon of Oil** (whole numbers) and **4:12 Super Hauler Truck** (fractions) are word problems involving multiplicative comparison. The building blocks of fractions are the topic in task **3:6 Unit Fraction Ideas**.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:7 Shipwrecks

Teacher Notes

Central math concepts

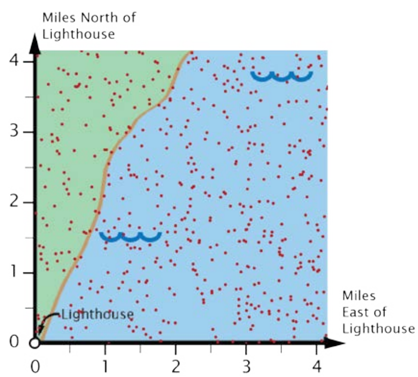
The coordinate plane is a relatively recent invention in the history of mathematics. Introduced in the 17th Century, this representation scheme was revolutionary for the way it connected algebra and geometry. Today the coordinate plane is an essential representation scheme for functions, two-variable equations, and bivariate data sets. Moreover, any time a computer is used for geometric or spatial applications, such as for GPS, robot vision, or finding shortest routes, those problems are being represented within the computer by means of numerical coordinates for the locations and boundary points of shapes. (A computer doesn't "see" a rectangle, but it can store and manipulate the rectangle in the form of a list of four ordered pairs of numbers.)

In task 5:7, the (x, y) coordinates of a point have a spatial representation as map coordinates. In later grades, where contexts will often involve a proportional relationship between two covarying quantities, the (x, y) coordinates of a point will often be a particular pair of values taken on by the quantities as they covary.

Connection to the number line. In a number line, every point in the continuum of the line is imagined to be labeled with a real number. In a coordinate plane, every point in the continuum of the plane is imagined to be labeled with an ordered pair of real numbers.

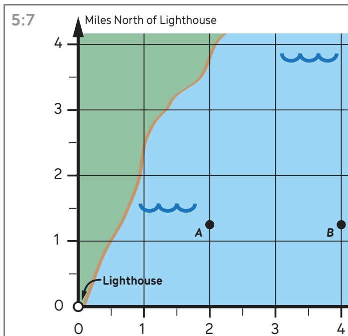
In a coordinate plane, the two coordinate axes are number lines. In contextual problems, each coordinate axis (number line) can be thought of as a measurement scale with a unit. In task 5:7, the unit for each number line is 1 mile, but in general the units on the two axes don't have to have the same value or even be quantities of the same type. For example, in a distance-time graph, the unit on the horizontal axis might be 1 hour while the unit on the vertical axis might be 1 mile.

Gridlines on a coordinate plane are the two-dimensional extension of the tick marks on a number line. Gridlines can make it easier to determine the coordinates of points by visual inspection. But it must be understood that a coordinate plane includes all of the plane's infinitely many points, not just the special points found at the intersections of gridlines (see figure). Likewise, a number line contains all of the line's infinitely many points, not just the points indicated by tick marks.



Dots show the locations of 500 randomly chosen points in the part of the coordinate plane used in task 5:7.

5:7

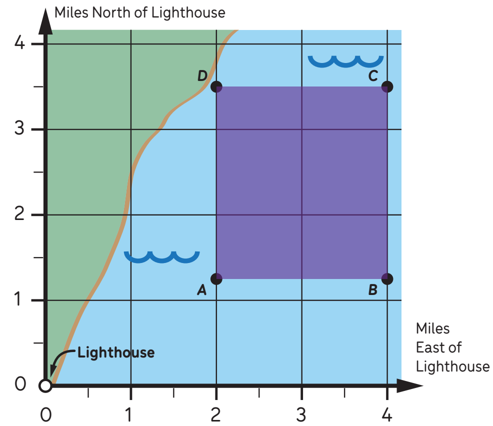


The map shows an ocean near a coastline.

Shipwrecks are at locations A $(2, 1\frac{1}{4})$ and B $(4, 1\frac{1}{4})$. Shipwrecks are also at locations C $(4, 3\frac{1}{2})$ and D $(2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

Answer

(1) See figure. **(2)** $4\frac{1}{2}$ mi^2.



Click here for a student-facing version of the task.

Refer to the Standards

5.NF.B.4b, 5.G.A; MP.2, MP.4, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

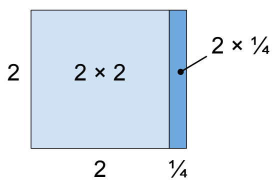
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: understanding the meaning of 1 square unit of area; relating number lines to quantities in context; and doubling a mixed number or fraction.

Extending the task

How might students drive the conversation further?

- If students determined the area of the shaded rectangle using multiple strategies or thought processes, students could connect those approaches to each other and to the rectangle area formula. According to the formula, the area in square miles equals the product $2 \times 2\frac{1}{4}$, or equivalently the product $2 \times \frac{9}{4}$.
    - Using distributive property thinking with the mixed number factor, the product $2 \times (2 + \frac{1}{4})$ equals the sum $2 \times 2 + 2 \times \frac{1}{4} = 4 + \frac{1}{2}$. The terms in this equation can be identified in an area diagram (see figure).

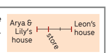


    - Alternatively, using unit thinking with the product $2 \times \frac{9}{4}$, one could make sense of the product $2 \times \frac{9}{4} = \frac{9}{2}$ by thinking, "I could double 9 fourths by replacing the unit of fourths with a unit twice as large, making 9 halves."
    - Of course, the fraction product algorithm $2 \times \frac{9}{4} = \frac{2}{1} \times \frac{9}{4} = \frac{18}{4}$ gives the result too.
- Students could draw additional tick marks on the coordinate axes to represent fourths of a mile, then add corresponding gridlines. What do students notice, wonder, or want to calculate about this picture? For example, how many of the resulting grid squares are in the shaded rectangle? What is the area of each of those grid squares in square miles? How do those numbers determine the area of the shaded rectangle?

Related Math Milestones tasks

5:6

5:6 **(1)** Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?



5:11

5:11 Juliet said, "I'm thinking of a rectangle. Its area is 1 square unit. Its perimeter is more than 1 million units."
(1) Is Juliet thinking of something possible or impossible? Use math to decide for sure.
(2) Explain your reasoning to your classmates. Revise your explanation based on suggestions from your classmates.

5:14

5:14 Brandon was reading his math book. He saw the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$. He said, "I don't get it—where did the 3 and the $\frac{3}{8}$ come from?" Write an explanation that could answer Brandon's question.

5:5

5:5 Write the requested values.

$4087 \times 53 = ?$	$\frac{1}{10} \div 10 = ?$	$0.4 \times 0.9 = ?$
$246 \times 914 = ?$	$\frac{7}{8} \times \frac{5}{3} = ?$	$0.75 \div 0.01 = ?$
$9744 \div 12 = ?$	$8 \times ? = 73$	$0.63 \div 0.3 = ?$
$1461 \div 6 = ?$	$3 \div \frac{1}{8} = ?$	$0.86 + 0.4 = ?$
$4 - (8 - 4) = ?$	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	$0.72 - 0.17 = ?$
	$\frac{1}{3} \div (6 \times 5) = ?$	$0.02 + 0.2 = ?$
		$0.8 - 0.55 = ?$
		$637 - 1.31 = ?$

Fraction products receive a conceptual emphasis in tasks **5:6 Corner Store**, **5:11 Juliet's Rectangle**, and **5:14 Brandon's Equation**, and a procedural emphasis in task **5:5 Calculating**.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

- The coordinate axes are shown as heavy lines in order to emphasize the connection to number lines.
- On the spatial scale of the problem (4 miles × 4 miles), objects the size of a shipwreck or a lighthouse are well modeled as single points.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:7? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:7? In what specific ways do they differ from 5:7?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:7 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

6:3

6:3 The table shows temperatures at the South Pole before and after midnight on October 10–11, 2019.

Time	Hours after Midnight	Temp °F
8:00 pm	−4	−42
9:00 pm	−3	−42
10:00 pm	−2	−41
11:00 pm	−1	−40
Midnight	0	−39
1:00 am	1	−39
2:00 am	2	−38

Plot the data on graph paper and label the plot. Describe any patterns you see.

6:4

6:4 My car drives 570 mi with 15 gal of gas. **(1)** *Mental math/Pencil and paper* **(a)** If I drive 57 mi, I'll use ___ gal. **(b)** If I drive 5,700 mi, I'll use ___ gal. **(c)** If I have 5 gal left, I can drive ___ more mi. **(d)** I can drive ___ mi with 30 gal. **(2)** *Calculator* Calculate both unit rates for the proportional relationship. **(3)** **(a)** If I drive 532 mi, I'll use ___ gal. **(b)** If I have 11 gal left, I can drive ___ more mi. **(4)** Make a two-column table using your answers to (1a), (1c), (1d), (3a), and (3b). Then use graph paper to plot the values in the table. Label the axes of your plot.

6:12

6:12 **(1)** What is the area of the triangle in the coordinate plane with vertices (1, 2), (−5, 2), and (−8, 9)? **(2)** How does the area change if we change the third vertex to (−3, 9)?

In later grades, task **6:3 South Pole Temperatures** involves using the coordinate plane to display a set of bivariate measurement data. Task **6:4 Gas Mileage** (part (4)) involves using the coordinate plane to plot values from a proportional relationship. Task **6:12 Coordinate Triangle** places a triangle in the coordinate plane.

4:13

4:13 **(1)** A red rectangle has length $L = 12$ in and width $W = 6$ in. Use the formula $A = L \times W$ to find the area of the red rectangle.

(2) A blue rectangle has length 1 ft and width $\frac{1}{2}$ ft. Draw a picture to show that two copies of the blue rectangle make one square foot. Based on your picture, what is the area of the blue rectangle?

(3) Do the red rectangle and the blue rectangle have equal areas? Tell how you decided.

4:5

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} = \underset{(a)}{\_\_\_}$

$6 \times \frac{4}{7} = \underset{(b)}{\_\_\_}$

$86 \times \frac{1}{86} = \underset{(c)}{\_\_\_}$

$6 \times \frac{8}{2} = \underset{(d)}{\_\_\_}$

$9 \times \frac{1}{9} = \underset{(e)}{\_\_\_}$

$9 \times \frac{2}{9} = \underset{(f)}{\_\_\_}$

(1g) Which answer is twice as much as the answer for (e)?

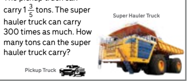
(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

4:12

4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?



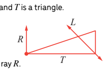
4:8

4:8 L is a line, R is a ray, and T is a triangle. True or false:

(1) Line L is a line of symmetry for triangle T.

(2) Line L intersects ray R.

(3) Triangle T has two angles measuring less than 90 degrees.



In earlier grades, task **4:13 Area Units** involves rectangle area in a case of a fractional length. Task **4:5 Fraction Products and Properties** involves multiplying a fraction by a whole number with a conceptual emphasis, and task **4:12 Super Hauler Truck** involves a multiplicative comparison leading to multiplying a fraction by a whole number in context. Task **4:8 Shapes with Given Positions** involves the distinction between a geometric (mathematical) object and the physical diagram that refers to it.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

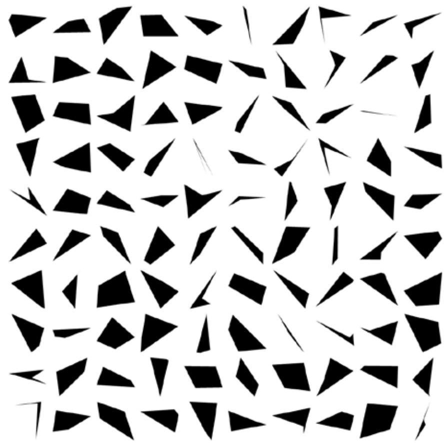
- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:8 Alana's New Shape Category

Teacher Notes

Central math concepts

Mathematics is a creative endeavor. Definitions can be invented, and then the consequences of those definitions can be discovered through mathematical reasoning. Anyone has the power to make mathematics!



The study of geometry tends to concentrate on figures with special properties, such as isosceles triangles, parallelograms, or regular polygons. These figures aren't representative of the larger classes to which they belong; for example, three points placed at random in the plane almost never define an isosceles triangle. Special figures are worth studying for many reasons, including the way their special features allow us to draw further conclusions about them. However, it is also worth acknowledging that these shapes are atypical. For example, the figure shows 100 quadrilaterals randomly generated by a computer. None of the quadrilaterals are squares, rhombuses, rectangles, parallelograms, or trapezoids.

All of them, however, are alana-gons!

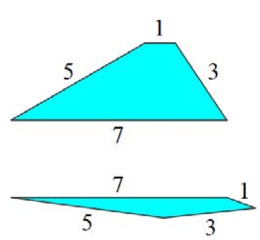
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: remembering definitions of special quadrilaterals; reasoning about figures based on attributes; drawing examples of special quadrilaterals; and basing mathematical arguments on figures.

Extending the task

How might students drive the conversation further?

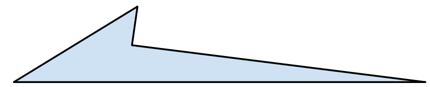
- Students could toss four crumpled-up balls of paper onto the floor to mark four locations, then connect the locations with string, cash register tape, or similar. Then check: is the resulting figure an alana-gon?
- Students could investigate whether there is a hexagon with all six sides of different lengths.
- Given the two alana-gons shown (which have equal perimeters of 16 units), students could decide by inspection which alana-gon has the greater area (it is the one on top in the figure).



5:8 A *scalene triangle* is a triangle in which the sides all have different lengths. Thinking about this, Alana decided there should also be a name for quadrilaterals in which the sides all have different lengths. She said, "I'll name them after myself." She defined an *alana-gon* to be a quadrilateral in which the four sides all have different lengths. **(1)** Draw an example of an alana-gon. **(2)** True or false: **(a)** All squares are alana-gons. **(b)** No trapezoids are alana-gons.

Answer

(1) Answers may vary; see example.
(2) (a) False. **(b)** False.



Click here for a student-facing version of the task.

Refer to the Standards

5.G.B; MP.3, MP.6, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

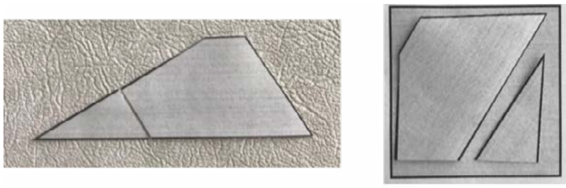
Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

It is intentional that the task tells the story of a student, Alana, rather than asking in the abstract about whether squares or trapezoids belong to an identified category. This is intended to emphasize the possibilities for creativity and invention in making mathematics.

- Students could try to draw a quadrilateral with the same perimeter as the top alana-gon, but with greater area. (The image shows one answer: a square with side length 4 units has greater area than the top alana-gon.)

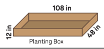


After cutting the alana-gon into two pieces as shown, both pieces fit in a square of side length 4 units, with area left over.

Related Math Milestones tasks

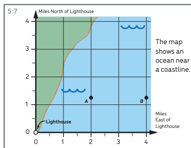
5:3

5:3 A neighborhood garden will have 6 wooden planting boxes. Every box will have the same shape (see diagram). Soil can be bought by the truckload; a truckload is 54 ft^3 of soil. How many truckloads of soil will fill all of the boxes?



5:7

5:7



The map shows an ocean near a coastline.

Shipwrecks are at locations $A\ (2, 1\frac{1}{4})$ and $B\ (4, 1\frac{1}{4})$. Shipwrecks are also at locations $C\ (4, 3\frac{1}{2})$ and $D\ (2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

5:11

5:11 Juliet said, "I'm thinking of a rectangle. Its area is 1 square unit. Its perimeter is more than 1 million units."

(1) Is Juliet thinking of something possible or impossible? Use math to decide for sure.

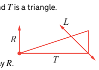
(2) Explain your reasoning to your classmates. Revise your explanation based on suggestions from your classmates.

6:12

6:12 **(1)** What is the area of the triangle in the coordinate plane with vertices $(1, 2)$, $(-5, 2)$, and $(-8, 9)$? **(2)** How does the area change if we change the third vertex to $(-3, 9)$?

4:8

4:8 L is a line, R is a ray, and T is a triangle. True or false:



(1) Line L is a line of symmetry for triangle T.

(2) Line L intersects ray R.

(3) Triangle T has two angles measuring less than 90 degrees.

Tasks **5:3 Neighborhood Garden**, **5:7 Shipwrecks**, and **5:11 Juliet's Rectangle** involve geometric measurement.

In later grades, task **6:12 Coordinate Triangle** places a geometric figure in the coordinate plane.

In earlier grades, task **4:8 Shapes with Given Positions** involves definitions of geometric figures and their measures.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:8? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:8? In what specific ways do they differ from 5:8?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:8 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

5:8 Alana's New Shape Category

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:9 Walkathon

Teacher Notes

Math Milestones™

Central math concepts

Based on interviews with middle-grades students,[†] education researcher Larry Sowder listed seven common strategies used by students to solve word problems. Ordered roughly from the least desirable to the most desirable, the strategies Sowder observed were as follows:

1. Find the numbers and add (or multiply or subtract...; the choice may be dictated by what has taken place in class recently or by what operation the student feels most competent at doing).
2. Guess at the operation to be used.
3. Look at the numbers; they will "tell" you which operation to use (e.g., "...if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and] 3, it looks like a division problem because of the size of the numbers").
4. Try all the operations and choose the most reasonable answer.
5. Look for isolated "key" words or phrases to tell which operations to use (e.g., "all together" means to add).
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.
7. Choose the operation whose meaning fits the story.

Especially when a word problem involves fractional quantities, the only robust strategy on Sowder's list is the last strategy: choosing the operation whose meaning fits the story. Single-step problems are useful for learning the core meanings and uses of the operations so that students can grow mathematically into using all four operations flexibly when solving multi-step problems. For multi-step problems, success in problem solving depends on understanding the meanings and uses of addition, subtraction, multiplication, and division, and the relationships between the operations. Bear in mind also that beyond grade 5, solving problems in middle grades will involve using algebra to calculate with variables as if they were numbers. For such problems, students will not be able to rely on looking at the numbers to "tell" them what operation to use. Understanding the meanings of the operations is therefore valuable as preparation not only for powerful problem solving in the elementary grades, but also as preparation for algebra.

In particular, task 5:9 involves addition and subtraction and the relationship between them. The mathematical relationship between addition and subtraction is that $C - A$ is the unknown addend in $A + ? = C$. (One might paraphrase this statement by saying that, "Given a total and one part, subtraction finds the other part.") Therefore, problems involving subtraction also implicitly involve addition, because subtraction finds an unknown addend. This is why a subtraction calculation is checked by adding.

5:9 On Saturday there was a walkathon.

Catherine

I walked $\frac{1}{3}$ mile farther than Leslie.

I walked $1\frac{1}{4}$ mile.

How many miles did Leslie walk?

Answer

$\frac{11}{12}$ mi.

Click here for a student-facing version of the task.

Refer to the Standards

5.NF.A.1, 2; MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application, Procedural skill and fluency

Additional notes on the design of the task

Word problems involving Compare situations can sometimes consist of complex text. Therefore, task 5:9 presents the given information in the form of a monologue by Catherine. This could also invite an approach of having students convey the task to each other by reciting the monologue.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:9? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:9? In what specific ways do they differ from 5:9?

From an abstract point of view, there's not much more to say than that. Educationally however, there is a lot more to say, because the mathematical relationship between addition and subtraction can play out in real life in conceptually distinct ways. In fact, there are three main meanings or uses for addition and subtraction:[‡]

- Adding To/Taking From
- Putting Together/Taking Apart
- Comparing

Furthermore, in a word problem, some quantities in the situation are known while others are initially unknown; the various possibilities for what is known and what is initially unknown combine with the main meanings of addition and subtraction to give a total of fifteen basic situation types for elementary addition and subtraction word problems.

In particular, the situation type in task 5:9 is called "Compare with Smaller Unknown." It is a Compare situation because Catherine is using subtraction to compare her distance with Leslie's distance; and more specifically, the situation is "Compare with Smaller Unknown" because the initially unknown quantity is how far Leslie walked (and Leslie walked the shorter distance). During the primary grades, students work with all situation types and all variations in the known and unknown quantities, with quantities given as whole numbers. In the upper-elementary grades, these understandings of addition and subtraction are extended into working with fractional quantities. Although the algorithms for performing calculations with fractions are different from those for performing base-ten calculations with whole numbers, the underlying meanings and uses of addition and subtraction are the same regardless of whether the numbers involved are whole numbers, fractions, decimals, or even variables.

The difference $1\frac{1}{4} - \frac{1}{3}$ could be calculated in many ways. All methods will involve using equivalent fractions to replace the given problem with one in which the denominators are equal (see figure). Transforming a problem to make it easier to solve is one of the most powerful mathematical practices of all.

$$\begin{array}{ccc} 1\frac{1}{4} & - & \frac{1}{3} \\ \shortparallel & & \shortparallel \\ \frac{15}{12} & - & \frac{4}{12} \end{array}$$

Adding and subtracting with unequal denominators need not be approached as an algorithmic process. For example, a student could rewrite the problem $1\frac{1}{4} - \frac{1}{3}$ as $\frac{5}{4} - \frac{1}{3}$ and then use fraction equivalence to replace $\frac{5}{4}$ with its equal $\frac{15}{12}$ and replace $\frac{1}{3}$ with its equal $\frac{4}{12}$; this gives $\frac{5}{4} - \frac{1}{3} = \frac{15}{12} - \frac{4}{12}$. Alternatively, a student could compare $\frac{1}{3}$ and $\frac{1}{4}$ in the equivalent forms $\frac{4}{12}$ and $\frac{3}{12}$, recognizing that $\frac{1}{3}$ is greater than $\frac{1}{4}$ by an amount $\frac{1}{12}$, and therefore the answer to the task is $1 - \frac{1}{12}$. Replacing the problem $1\frac{1}{4} - \frac{1}{3}$ with the equivalent problem $1 - \frac{1}{12}$ is analogous to a mental calculation of a difference in a case like 1,002 - 7, in which case we could think of the equivalent problem 1,000 - 5.

Word problems vary considerably in the uses to which they put the basic operations, and they also vary in the complexity of the calculation

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:9 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Sowder, Larry. (1988). *Concept-Driven Strategies for Solving Problems in Mathematics. Final Project Report*. San Diego State Univ., CA. Center for Research in Mathematics and Science Education. https://files.eric.ed.gov/fulltext/ED290629.pdf

‡ See Table 2, p. 9 of *Progressions for the Common Core State Standards in Mathematics (draft): K, Counting and Cardinality; K–5, Operations and Algebraic Thinking* (Common Core Standards Writing Team, May 29, 2011. Tucson, AZ: Institute for Mathematics and Education, University of Arizona).

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

required to obtain a final numerical answer. The overall challenge of a word problem depends on both the situational complexity and the computational complexity. Students for whom the calculation $1\frac{1}{4} - \frac{1}{3}$ is time-consuming and/or effortful may need to be redirected to the context after obtaining the result $1\frac{1}{4} - \frac{1}{3} = \frac{11}{12}$, so as to relate the numbers in this equation to the context and answer the question in the task.

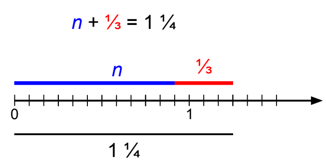
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: generating equivalent fractions; subtracting fractions with equal denominators; working with mixed numbers; using number sense of fraction size; and representing addition and subtraction on a number line.

Extending the task

How might students drive the conversation further?

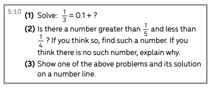
- Students could be asked to write an equation model for the situation, such as $n + \frac{1}{3} = 1\frac{1}{4}$. The equation could be represented on a number line, as shown in the figure.



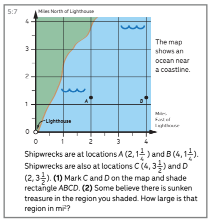
- Students could check the reasonableness of their answers, or predict the approximate value of the answer, by estimating that the answer will be close to 1 mi since $\frac{1}{3}$ and $\frac{1}{4}$ are close in value.
- Students could be asked to imagine that the path for the walkathon had been marked with flags every twelfth of a mile. In that case, how many flags did Catherine pass? How many flags did Leslie pass? How many more flags did Catherine pass than Leslie? These answers could be used to create a new version of the problem (keeping all the distances the same): "I passed ______ more flags than Catherine. I passed ______ flags. How many flags did Leslie pass?"

Related Math Milestones tasks

5:10

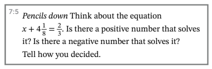

5:10 **(1)** Solve: $\frac{1}{3} = 0.1 + ?$
(2) Is there a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$? If you think so, find such a number. If you think there is no such number, explain why.
(3) Show one of the above problems and its solution on a number line.

5:7

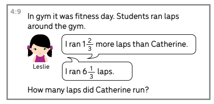


Shipwrecks are at locations $A\ (2, 1\frac{1}{4})$ and $B\ (4, 1\frac{1}{4})$. Shipwrecks are also at locations $C\ (4, 3\frac{1}{2})$ and $D\ (2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi²?

7:5


7:5 *Pencils down* Think about the equation $x + 4\frac{1}{8} = \frac{2}{3}$. Is there a positive number that solves it? Is there a negative number that solves it? Tell how you decided.

4:9


4:9 In gym it was fitness day. Students ran laps around the gym.

How many laps did Catherine run?

Task **5:10 Number System, Number Line** (part (1)) features an unknown addend problem involving both a fraction and a decimal. Task **5:7 Shipwrecks** involves finding the difference between $3\frac{1}{2}$ and $1\frac{1}{4}$.

In later grades, task **7:5 Is There a Solution? (Addition)** involves an equation for an unknown addend in the rational numbers.

In earlier grades, task **4:9 Fitness Day** is a word problem of situation type “Compare with Smaller Quantity Unknown” in which the calculation is similar to the one in task 5:9, except with equal denominators.

5:9 Walkathon

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:10 Number System, Number Line

Teacher Notes

Central math concepts

In mathematics and science, two powerful conceptions of number live side by side. In one conception, a real number is a label for a point on the number line. And in a related but distinct conception, a real number is a possible magnitude of some quantity measured with a chosen unit. Both conceptions of number are learned as early as grade 3.[†] Importantly, in neither conception is the *format* of the number essential: we can label points on the number line with fractions like $\frac{3}{4}$, or we can label them with multi-digit numbers like 0.75. And we can express a length measurement as $\frac{1}{5}$ mile, or we can express it as 0.2 mile.

Just as numbers can be understood without reference to their format, so can operations be understood independently of the format of the numbers on which they operate. For example, subtraction is an operation of separating or taking-from, regardless of whether the quantities being separated or decreased are written as fractions or as multi-digit symbols using base-ten.

Calculation algorithms, on the other hand, do depend on a number's format. The algorithms for fractions and multi-digit numbers are very different, yet the core meanings of division (quotative and partitive) are the same regardless of the format of the numbers involved.

> 6:14 *Pencil and paper* **(1)** $81.53 \div 3.1 = ?$ **(2)** $\frac{7}{8} \div \frac{2}{3} = ?$ **(3)** Check both of your answers by multiplying.

Task **6:14 Dividing Decimals and Fractions**. The division operation has the same meaning in parts (1) and (2), but the calculation algorithm depends on the number format in each case.

Because calculation algorithms differ so greatly depending on the format of the numbers, curriculum lessons and units often concentrate separately on fractions and decimals. But too hard a wall of separation risks obscuring fundamental concepts about both numbers and operations. In part (1) of task 5:10, the unknown number is $0.1 - \frac{1}{3}$ because $\frac{1}{3} = 0.1 + ?$ is an unknown addend problem. The idea of subtraction as an unknown addend problem is an idea so elementary that kindergarten students work with it, and so profound that students will continue to rely on it throughout their work with fractions and decimals, rational numbers, real numbers, algebraic expressions, complex numbers, and matrices.

Equally profound are the concepts of number that are conveyed by the metaphor of the number line. As a geometric line, the number line is infinitely dense with points, meaning that between any two points there are infinitely many points; therefore, given any two numbers, there are infinitely many numbers with values in between. The measurement conception of a number agrees with that idea too, because given any unit, no matter how small, one can always imagine partitioning it and using one of the parts as a new, smaller unit. This smaller unit can then be iterated on the number line with no gaps or overlaps. For example, with respect to part (3) of task 5:10, we could iterate the unit $\frac{1}{40}$ on the number

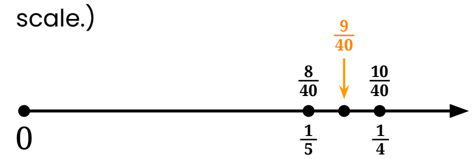
> 5:10 **(1)** Solve: $\frac{1}{3} = 0.1 + ?$
> **(2)** Is there a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$? If you think so, find such a number. If you think there is no such number, explain why.
> **(3)** Show one of the above problems and its solution on a number line.

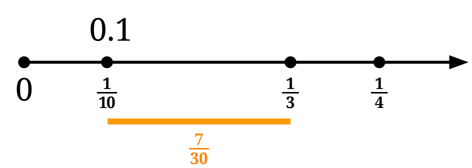
Answer

(1) $\frac{7}{30}$. (Note: the repeating decimal answer $0.2\overline{3}$ isn't necessarily expected at this grade, but it is also correct.)

(2) There is a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$. Examples may vary: for example, $\frac{9}{40}$, $\frac{2}{9}$, $\frac{21}{100}$, $= 0.21$, $\frac{22}{100} = \frac{11}{50} = 0.22$, $\frac{23}{100} = 0.23$, $\frac{24}{100} = \frac{6}{25} = 0.24$, $\frac{1}{5} + \frac{1}{1{,}000{,}000} = 0.200001$, $\frac{1}{4} - \frac{1}{1{,}000{,}000} = 0.249999$.

(3) See the figures for two possibilities. (Note: each figure is drawn approximately to scale, but the two figures are not drawn to the same scale.)





Click here for a student-facing version of the task.

Refer to the Standards

5.NF.A.1; MP.1, MP.3, MP.5. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

line and thereby represent the magnitudes $\frac{8}{40}, \frac{9}{40}, \frac{10}{40}$, in particular representing a magnitude $\frac{9}{40}$ that is intermediate between $\frac{8}{40} = \frac{1}{5}$ and $\frac{10}{40} = \frac{1}{4}$.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: renaming fractions with denominators of 10 to decimals; understanding subtraction as an unknown addend problem; generating equivalent fractions; number sense of fraction sizes; and using a number line to support and communicate mathematical reasoning.

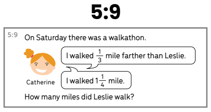
Extending the task

How might students drive the conversation further?

- Finding a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$ might lead students to wonder whether it is the case that between any two different numbers there is always another number. Exploring this question could lead to a more general insight: between any two different numbers there are always infinitely many other numbers.
- Students could make sense of parts (1) and (2) by creating a word problem for each. For example, "A. ran $\frac{1}{4}$ mile. B. ran $\frac{1}{5}$ mile. I didn't run as far as A., but I ran farther than B. What is a distance I could have run?"

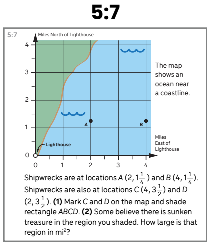
Related Math Milestones tasks

5:9


5:9 On Saturday there was a walkathon.

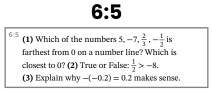
How many miles did Leslie walk?

5:7

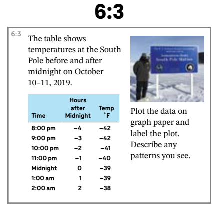


Shipwrecks are at locations $A\ (2, 1\frac{1}{4})$ and $B\ (4, 1\frac{1}{4})$. Shipwrecks are also at locations $C\ (4, 3\frac{1}{2})$ and $D\ (2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

6:5


6:5 **(1)** Which of the numbers $5, -7, \frac{2}{3}, -\frac{1}{2}$ is farthest from 0 on a number line? Which is closest to 0? **(2)** True or False: $\frac{1}{2} > -8$. **(3)** Explain why $-(-0.2) = 0.2$ makes sense.

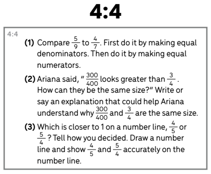
6:3


6:3 The table shows temperatures at the South Pole before and after midnight on October 10–11, 2019.

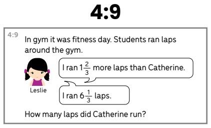
Time	Hours after Midnight	Temp °F
8:00 pm	−4	−42
9:00 pm	−3	−42
10:00 pm	−2	−41
11:00 pm	−1	−40
Midnight	0	−39
1:00 am	1	−39
2:00 am	2	−38

Plot the data on graph paper and label the plot. Describe any patterns you see.

4:4


4:4
(1) Compare $\frac{5}{9}$ to $\frac{4}{7}$. First do it by making equal denominators. Then do it by making equal numerators.
(2) Ariana said, "$\frac{300}{400}$ looks greater than $\frac{3}{4}$. How can they be the same size?" Write or say an explanation that could help Ariana understand why $\frac{300}{400}$ and $\frac{3}{4}$ are the same size.
(3) Which is closer to 1 on a number line, $\frac{4}{5}$ or $\frac{5}{4}$? Tell how you decided. Draw a number line and show $\frac{4}{5}$ and $\frac{5}{4}$ accurately on the number line.

4:9


4:9 In gym it was fitness day. Students ran laps around the gym.

How many laps did Catherine run?

Task **5:9 Walkathon** is a word problem for which the equation model could be an unknown-addend problem with a fraction and a mixed number. Task **5:7 Shipwrecks** involves the extension of the number line to the coordinate plane.

In later grades, task **6:5 Positive and Negative Numbers** emphasizes understanding a rational number as a point on the number line, and task **6:3 South Pole Temperatures** involves using the coordinate plane to plot a data set with observations that include signed numbers.

In earlier grades, task **4:4 Comparing Fractions with Equivalence** involves ideas of fraction magnitude and the number line. Task **4:9 Fitness Day** is a word problem for which the equation model could be an unknown-addend problem with mixed numbers (equal denominators).

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

The task rewards persistence, in the sense that parts (1) and (2) both present initial challenges. In part (1), the fraction $\frac{1}{3}$ cannot be written as a terminating decimal. In part (2), it is natural to begin by finding the least common denominator of $\frac{1}{4}$ and $\frac{1}{5}$, but that choice of denominator results in consecutive numerators (5 and 4).

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:10? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:10? In what specific ways do they differ from 5:10?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:10 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit*

† See domain 3.NF in the Standards, when students 'understand a fraction as a number on the number line' and also 'define the interval from 0 to 1 as the whole and partition it into b equal parts,' 'recognizing that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.' (Single quotation marks indicate paraphrase; see the exact language online.)

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:11 Juliet's Rectangle

Teacher Notes

Math Milestones™

Central math concepts

If you ask someone to draw a rectangle, they will probably draw a rectangle that is just a little wider than it is tall. However, rectangles can be any shape that satisfies the mathematical definition. True, it would probably be impossible to draw Juliet's rectangle using physical length units; for example, if the length unit were 1 foot, then Juliet's rectangle would be no wider than a bacterium, and over 90 miles long! Nevertheless, mathematical rectangles exist with any positive numbers as dimensions, and we may think about these rectangles and draw mathematical conclusions about them.

One mathematical conclusion we might draw about Juliet's rectangle is that the side-length measurements can't both be whole numbers. That's because the only whole-number factors with a product of 1 are the factors 1 and 1, but the perimeter of a 1-by-1 rectangle is 4 units, too small to meet Juliet's perimeter condition. Thus, Juliet's rectangle takes us beyond whole numbers, potentially raising questions about the meaning of multiplying by a fraction.

In general, the product $(\frac{a}{b}) \times q$ is a parts of a partition of q into b equal parts, equivalently the result of a sequence of operations $a \times q \div b$ (CCSS 5.NF.B.4). So for example, in Juliet's rectangle, suppose we take the length to be 500,000 units, which will certainly satisfy the perimeter requirement. Because the rectangle has area 1 square unit, the width of the rectangle is the unknown factor in ? × 500,000 = 1. Therefore, the unknown factor is 1 ÷ 500,000, and that quantity is 1 part in a partition of 1 into 500,000 equal parts, or $\frac{1}{500{,}000}$. To check this quotient, we can multiply. Doing so, $\frac{1}{500{,}000} \times 500{,}000$ is 1 part in a partition of 500,000 into 500,000 parts, and that's 1. Alternatively, $\frac{1}{500{,}000} \times 500{,}000$ is the result of the sequence of operations 1 × 500,000 ÷ 500,000, which is 1.

In Juliet's rectangle, the side-length measurements L and W satisfy the condition that $L \times W = 1$. This makes the numbers L and W an illustration of an important property of operations, the existence of multiplicative inverses. Whatever nonzero value W has, L is its multiplicative inverse, and vice versa.

Extending multiplication and division from whole numbers to fractions is perhaps the most important mathematical progression of the upper-elementary grades. This progression involves a substantial evolution in students' concepts about numbers and operations. The need to support students in making that conceptual evolution raises important questions: What aspects of earlier thinking about multiplying whole numbers will remain helpful when making sense of a product involving fractions? What new ways of thinking will be helpful? And what mathematical representations introduced during whole-number multiplication work are best suited to supporting the transition from multiplying and dividing whole numbers to multiplying and dividing all numbers?

5:11 Juliet said, "I'm thinking of a rectangle. Its area is 1 square unit. Its perimeter is more than 1 million units."

(1) Is Juliet thinking of something possible or impossible? Use math to decide for sure.

(2) Explain your reasoning to your classmates. Revise your explanation based on suggestions from your classmates.

Answer

(1) It is possible for a rectangle to have an area of 1 square unit and a perimeter of more than 1 million units. For example, the rectangle might have length $\frac{1}{1{,}000{,}000}$ unit and width 1,000,000 units. Then the area is 1 square unit, and the perimeter is $2{,}000{,}000 + \frac{2}{1{,}000{,}000}$ units. A distinct kind of argument could establish logically that Juliet's rectangle is possible without actually constructing a specific example of one; for example, it could be argued that "If we make the length be 1 divided by the width, then the width could be very large, like over 1 million units, but the area would still be 1 square unit." **(2)** Answers may vary but should include a noticeable revision toward such improvements as more complete and precise language, adding a labeled diagram, redrawing a diagram for greater clarity, etc.

Click here for a student-facing version of the task.

Refer to the Standards

5.NF.B; MP.1–3, MP.6, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: using large whole numbers; thinking about length and area units; working with unit fractions; multiplying by a unit fraction; and basing multiplicative reasoning on math diagrams.

Extending the task

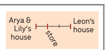
How might students drive the conversation further?

- Students could generate several different rectangles that satisfy Juliet's area and perimeter conditions and then look for patterns. For example, is 500,000 units the smallest possible length that Juliet's rectangle could have? Is there a largest possible length that Juliet's rectangle could have?
- Students could make up their own versions of Juliet's rectangle, such as, "I am thinking of a rectangle with area 12 square units and perimeter greater than 24 units. What might be the dimensions of my rectangle?"

Related Math Milestones tasks

5:6

5:6 **(1)** Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?

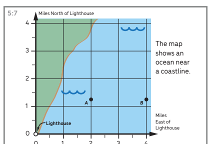


5:14

5:14 Brandon was reading his math book. He saw the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$. He said, "I don't get it—where did the 3 and the $\frac{3}{8}$ come from?" Write an explanation that could answer Brandon's question.

5:7

5:7



The map shows an ocean near a coastline.

Shipwrecks are at locations A $(2, 1\frac{1}{4})$ and B $(4, 1\frac{1}{4})$. Shipwrecks are also at locations C $(4, 3\frac{1}{2})$ and D $(2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

5:5

5:5 Write the requested values.

$4087 \times 53 = ?$	$\frac{1}{10} \div 10 = ?$	$0.4 \times 0.9 = ?$
$246 \times 914 = ?$	$\frac{7}{8} \times \frac{5}{3} = ?$	$0.75 \div 0.01 = ?$
$9744 \div 12 = ?$	$8 \times ? = 73$	$0.63 \div 0.3 = ?$
$1461 \div 6 = ?$	$3 \div \frac{1}{8} = ?$	$0.86 + 0.4 = ?$
$4 - (8 - 4) = ?$	$\frac{1}{2} + \frac{1}{3} - \frac{1}{5} = ?$	$0.72 - 0.17 = ?$
	$\frac{1}{3} \div (6 \times 5) = ?$	$0.02 + 0.2 = ?$
		$0.8 - 0.55 = ?$
		$637 - 1.31 = ?$

Fraction products receive a conceptual emphasis in tasks **5:6 Corner Store** and **5:14 Brandon's Equation**, an application emphasis in task **5:7 Shipwrecks**, and a procedural emphasis in task **5:5 Calculating**.

6:1

6:1 $\frac{2}{3}$ of a charging cord is $\frac{1}{2}$ meter long. How long is the charging cord? (Answer in meters.)

6:9

6:9 How much of a $\frac{3}{4}$-ton truckload is $\frac{2}{3}$ ton of gravel?

6:13

6:13 *Pencils down* Think about the equation $241p = \frac{3}{4}$. Is there a whole number that solves it? Is there a non-whole number that solves it? Convince a classmate that your answers are right.

In later grades, tasks **6:1 Charging Cord** and **6:9 Truckload of Gravel** are unknown-factor problems involving fractions. Task **6:13 Is There a Solution? (Multiplication)** is like Juliet's Rectangle in that the product of a whole number with an unknown number results in a product less than the whole number.

Additional notes on the design of the task

Note that the perimeter of the rectangle is only required to be "more than 1 million units"; the perimeter doesn't have to *equal* 1 million units.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:11? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:11? In what specific ways do they differ from 5:11?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:11 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

4:13

4:13 **(1)** A red rectangle has length $L = 12$ in and width $W = 6$ in. Use the formula $A = L \times W$ to find the area of the red rectangle.

(2) A blue rectangle has length 1 ft and width $\frac{1}{2}$ ft. Draw a picture to show that two copies of the blue rectangle make one square foot. Based on your picture, what is the area of the blue rectangle?

(3) Do the red rectangle and the blue rectangle have equal areas? Tell how you decided.

4:5

4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

$4 \times \frac{1}{7} = \underline{\quad}$ (a)

$6 \times \frac{4}{7} = \underline{\quad}$ (b)

$86 \times \frac{1}{86} = \underline{\quad}$ (c)

$6 \times \frac{8}{2} = \underline{\quad}$ (d)

$9 \times \frac{1}{9} = \underline{\quad}$ (e)

$9 \times \frac{2}{9} = \underline{\quad}$ (f)

(1g) Which answer is twice as much as the answer for (e)?

(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

4:12

4:12 The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

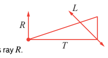
4:8

4:8 L is a line, R is a ray, and T is a triangle. True or false:

(1) Line L is a line of symmetry for triangle T.

(2) Line L intersects ray R.

(3) Triangle T has two angles measuring less than 90 degrees.



In earlier grades, task **4:13 Area Units** involves rectangle area in a case of a fractional length. Task **4:5 Fraction Products and Properties** involves multiplying a fraction by a whole number with a conceptual emphasis, and task **4:12 Super Hauler Truck** involves a multiplicative comparison leading to multiplying a fraction by a whole number in context. Task **4:8 Shapes with Given Positions** involves the distinction between a geometric (mathematical) object and the physical diagram that refers to it.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:12 Rain Measurements

Teacher Notes

Central math concepts

Students' data work in the upper-elementary grades concentrates on measurement data displayed in line plots.[†] In a line plot like the one shown in task 5:12, the "x" symbols are the individual data points.[‡]

The number line diagram in a line plot corresponds to the scale on the measurement tool that was used to generate the data. In task 5:12, the measurements are liquid volumes in liters, but if the measurements had been masses, for example, then the units on the scale would be kilograms or another unit of mass.

As for the vertical scale on a line plot, a vertical scale isn't shown on the line plot in task 5:12, but if a vertical scale had been shown, then it would be a count scale, meaning that the tick marks on the vertical scale would be the numbers 0, 1, 2, 3, and so on, indicating the number of observations above each tick mark.

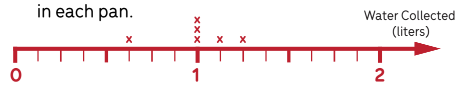
Interpreting a line plot involves grasping the correspondence between an "x" symbol or dot, its horizontal position on the measurement scale, and what fact about the situation is being thereby recorded. For example, the leftmost "x" symbol records the fact that one of the baking pans set out by the teacher collected $\frac{5}{8}$ of a liter of water.

From the individual data points (the "x" symbols), students can count to find the total number of observations, 6, which is one numerical summary of the data—and also the number of pans set out by the teacher.

There are close connections in every elementary grade between students' data work and their expanding use of numbers and operations in context; see Table 1, p. 4 of the relevant *Progression* document for a list of these connections in grades K–5. Students' work in representing and analyzing measurement data connects directly to their growing number sense of fractions and to their increasing ability to operate with fractions to solve problems in context. In task 5:12, students use a number line diagram marked in eighths. They use addition to determine the total volume of water collected, in liters. And they use division to determine the answer to the unknown factor problem $6 \times ? = 6$, where the factor on the left-hand side corresponds to the number of baking pans, the result on the right corresponds to the number of liters in all, and the unknown factor is the number of liters in one pan.

As the *Guidelines for Assessment and Instruction in Statistics Education Report* notes, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning."[§] Thus as the *Progression* document notes, "students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the context they represent" (p. 3).

5:12 Before it rained, the teacher went outside and placed identical baking pans on the ground. After it rained, the teacher brought the pans inside, and students measured how much water was collected in each pan.



If all the water collected were shared equally among the pans, how much water would be in each pan?

Answer

1 liter.

Click here for a student-facing version of the task.

Refer to the Standards

5.MD.B; MP.2, MP.4. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

- The answer to task 5:12, 1 liter, may draw attention to the fact that half of the observations have that exact value, while two values are slightly greater and another value is substantially less, so that 1 liter is a plausible "balance point" of the data; this prefigures work with the mean as a measure of center in grade 6.
- In an idealized rain shower, one might expect every baking pan to contain the same amount of water. However, there can also be variability in the rain collection and in the measurement process.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: thinking about capacity in units of liters; using number lines with fractions; calculating sums of mixed numbers with equal denominators; and using addition and division to solve problems in context.

Extending the task

How might students drive the conversation further?

- Students could consider what would have happened if the rain shower had lasted twice as long. What might the line plot have looked like?
- Students could use websites such as weather.gov to research the measurement of rain in weather research.

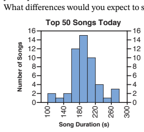
Related Math Milestones tasks

5:2

5:2 After a hurricane, the 12 residents of a nursing home didn't have any clean water to drink. Their neighbors donated 40 gallons of bottled water, which would provide ____ gallons for each resident.

6:7

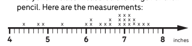
6:7 **(1)** Look up the 50 top songs on a music streaming service. Type each song's duration into a spreadsheet. **(2)** Write a sentence about the data giving a measure of center and a measure of variability. **(3)** Make a histogram of the data.* **(4)** Write a sentence describing the overall pattern of the distribution and any striking deviations from the overall pattern. **(5)** Imagine that one year from now, you go back online and repeat (1)–(4). In what ways would you expect the data distribution to look similar? What differences would you expect to see?



*Use this histogram for (4) and (5) if you don't do (3).

4:3

4:3 Everyone in class measured the length of their pencil. Here are the measurements:



(1) How many pencils were measured?

(2) How much longer was the longest pencil than the shortest pencil?

(3) Could two of the pencils be laid end to end to make a total length of 1 foot?

Like task 5:12, task **5:2 Water Relief** involves a division of an amount of a continuous measurement quantity by a whole number.

In later grades, measures of center are part of task **6:7 Song Length Distribution** (note that task 5:12 involves equally redistributing the water in the baking pans, which prefigures the statistical mean).

In earlier grades, task **4:3 Pencil Data** involves contextual problem solving based on interpreting data represented on a line plot.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:12? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:12? In what specific ways do they differ from 5:12?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:12 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2011, June 20). *Progressions for the Common Core State Standards in Mathematics (draft): K–3, Categorical Data; Grades 2–5, Measurement Data.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ In line plots generated by technology, data points are often marked by small filled circles, or "dots." Apart from that cosmetic feature, the terms *line plot* and *dot plot* are synonymous.

§ The *Guidelines for Assessment and Instruction in Statistics Education Report* was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

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Anticipating and responding to student thinking about the task

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- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:13 Frozen Yogurt Machine

Teacher Notes

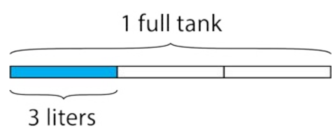
Central math concepts

Step 1. Task 5:13 is typical of multi-step word problems in that a quantity worth knowing in the situation isn't named in the text of the problem. That quantity is the number of liters in the machine when it is full. The value of this quantity isn't given—it also isn't asked for—yet the value is helpful for solving the problem that is posed. Finding this value involves thinking about unit fractions and division.

As students extend their understanding of multiplication and division from whole numbers to fractions, one idea that endures is the idea that division finds an unknown factor: $C \div A$ is the unknown factor in $A \times ? = C$. In task 5:13, the number of liters in the machine when it is full is initially an unknown factor, because $\frac{1}{3}$ of this number is 3:

$\frac{1}{3} \times ? = 3$.

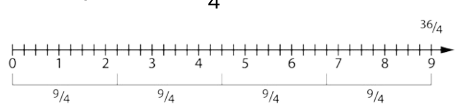
The unknown factor is 9, so that $3 \div \frac{1}{3} = 9$. See the figure for a representation of the unknown factor problem $\frac{1}{3} \times ? = 3$.



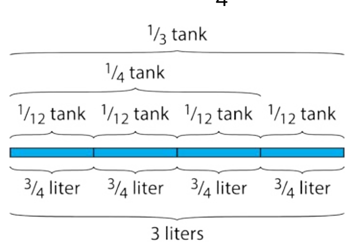
The figure also suggests more generally why dividing by a unit fraction $\frac{1}{n}$ amounts to multiplying by its reciprocal, n. For an illuminating discussion of fraction division with numerous diagrams and examples, see the 2017 series of blog posts by William McCallum and Kristin Umland on MathematicalMusings.org (part 1, part 2, part 3, part 4). In particular, part 3 concentrates on the reason why "dividing by a unit fraction is the same as multiplying by its reciprocal."

Step 2. Knowing there are 9 liters in the machine when it is full, the quantity of yogurt in the machine, in liters, when the machine is $\frac{1}{4}$ full is given by the product $\frac{1}{4} \times 9$. One-fourth of 9 means 1 part of a partition of 9 into 4 equal parts, or equivalently $9 \div 4$, and this quotient is $\frac{9}{4}$.

Partitioning 9 into 4 equal parts could also be accomplished by thinking about units and fraction equivalence, as shown in the diagram. First rewrite the number 9 as a fraction, $\frac{36}{4}$. This fraction is 36 parts of size one-fourth. When 36 parts of size one-fourth are partitioned into 4 equal parts, 1 of the parts will consist of 9 parts of size one-fourth; and 9 parts of size one-fourth are $\frac{9}{4}$.



Another way. (See the diagram.) When the yogurt in the machine decreases from $\frac{1}{3}$ of a tank to $\frac{1}{4}$ of a tank, it decreases by $\frac{1}{12}$ of a tank (because $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$). That $\frac{1}{12}$ of a tank is $\frac{1}{4}$ of the 3 liters, because $\frac{1}{12}$ is $\frac{1}{4}$ of $\frac{1}{3}$. And $\frac{1}{4}$ of 3 liters is $\frac{3}{4}$ ℓ. So when the yogurt in the machine decreases from $\frac{1}{3}$ full to $\frac{1}{4}$ full, it decreases by $\frac{3}{4}$ ℓ of yogurt. This gives the answer to the problem as $3 - \frac{3}{4} = 2\frac{1}{4}$ liters.



5:13 In a snack shop there is a frozen yogurt machine. When there is 3 *l* of frozen yogurt in the machine, the machine is $\frac{1}{3}$ full. How much frozen yogurt is in the machine when it is $\frac{1}{4}$ full?

Answer

$\frac{9}{4}$ ℓ, $2\frac{1}{4}$ ℓ, or 2.25 ℓ.

Click here for a student-facing version of the task.

Refer to the Standards

5.NF.B.6, 7; MP.1, MP.2, MP.4, MP.5, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

One approach to the problem would involve division of a whole number by a unit fraction, followed by multiplication of a whole number by a unit fraction; this approach could be expressed by the calculation $\frac{1}{4} \times (3 \div \frac{1}{3})$. However, students' thought processes might correspond instead to equivalent expressions, such as $(3 \times 3) \div 4$.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:13? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:13? In what specific ways do they differ from 5:13?

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: multiplying by a unit fraction; applying ideas of times-as-much in context; and basing multiplicative reasoning on math diagrams.

Extending the task

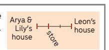
How might students drive the conversation further?

- Knowing that the machine holds $2\frac{1}{4}$ ℓ of frozen yogurt when it is $\frac{1}{4}$ full, students could observe that decreasing from $\frac{1}{3}$ full to $\frac{1}{4}$ full corresponded to a decrease of $3 - 2\frac{1}{4} = \frac{3}{4}$ ℓ of frozen yogurt. What fraction of a full tank is $\frac{3}{4}$ ℓ ? How can that fraction be seen in the situation?
- A student who approached the problem in one way (or drew one kind of diagram) could present their thinking to a partner who approached it a different way (or drew another kind of diagram). The partner could say back what the first student said. The two students could refine their explanations and responses until the mathematics has been effectively communicated.

Related Math Milestones tasks

5:6

5:6 **(1)** Arya and Lily's house is $\frac{1}{5}$ mile from the store. **(a)** Arya ran $\frac{1}{3}$ of the way from her house to the store. How far, in miles, did Arya run? **(b)** Lily ran $\frac{2}{3}$ of the way from her house to the store. How far, in miles, did Lily run? **(2)** It is $\frac{2}{5}$ mile from Leon's house to the store. **(a)** Leon ran $\frac{1}{3}$ of the way from his house to the store. How far, in miles, did Leon run? **(b)** Compare how far Leon and Lily ran; what do you notice, and why is it true?



5:2

5:2 After a hurricane, the 12 residents of a nursing home didn't have any clean water to drink. Their neighbors donated 40 gallons of bottled water, which would provide ____ gallons for each resident.

5:11

5:11 Juliet said, "I'm thinking of a rectangle. Its area is 1 square unit. Its perimeter is more than 1 million units."
(1) Is Juliet thinking of something possible or impossible? Use math to decide for sure.
(2) Explain your reasoning to your classmates. Revise your explanation based on suggestions from your classmates.

Task **5:6 Corner Store** involves fraction products in context. Task **5:2 Water Relief** involves interpreting the quotient of two whole numbers as a fraction. Task **5:11 Juliet's Rectangle** could promote thinking about the extension of multiplication from whole numbers to fractions.

6:1

6:1 $\frac{2}{3}$ of a charging cord is $\frac{1}{2}$ meter long. How long is the charging cord? (Answer in meters.)

6:9

6:9 How much of a $\frac{3}{4}$-ton truckload is $\frac{2}{3}$ ton of gravel?

6:6

6:6 A farmer uses a tractor to plant corn quickly in the springtime. The farmer plants 216 acres every 12 hours. Create a formula for the number of acres the farmer plants in n hours.

6:13

6:13 *Pencils down* Think about the equation $241p = \frac{3}{4}$. Is there a whole number that solves it? Is there a non-whole number that solves it? Convince a classmate that your answers are right.

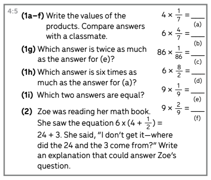
In later grades, tasks **6:1 Charging Cord** and **6:9 Truckload of Gravel** are word problems that use multiplication and division operations on fractions, completing the extension of arithmetic from whole numbers to fractions. Task **6:6 Planting Corn** is about proportionality (which can involve application of times as much thinking). In task **6:13 Is There a Solution? (Multiplication)**, the equation can be made sense of as a question about multiplicative scaling.

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:13 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

4:5



4:5

(1a–f) Write the values of the products. Compare answers with a classmate.

(1g) Which answer is twice as much as the answer for (e)?

(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

$4 \times \frac{1}{7} = \underline{\quad}$ (a)

$6 \times \frac{4}{7} = \underline{\quad}$ (b)

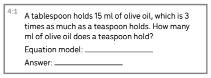
$86 \times \frac{1}{86} = \underline{\quad}$ (c)

$6 \times \frac{8}{2} = \underline{\quad}$ (d)

$9 \times \frac{1}{9} = \underline{\quad}$ (e)

$9 \times \frac{2}{9} = \underline{\quad}$ (f)

4:1



4:1

A tablespoon holds 15 ml of olive oil, which is 3 times as much as a teaspoon holds. How many ml of olive oil does a teaspoon hold?

Equation model: ____________

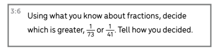
Answer: ____________

4:12

4:12

The pickup truck can carry $1\frac{3}{5}$ tons. The super hauler truck can carry 300 times as much. How many tons can the super hauler truck carry?

3:6



3:6

Using what you know about fractions, decide which is greater, $\frac{1}{73}$ or $\frac{1}{41}$. Tell how you decided.

In earlier grades, task **4:5 Fraction Products and Properties** can involve times-as-much thinking and sense-making about fraction products. Tasks **4:1 Tablespoon of Oil** (whole numbers) and **4:12 Super Hauler Truck** (fractions) are word problems involving multiplicative comparison. The building blocks of fractions are the topic in task **3:6 Unit Fraction Ideas**.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

5:14 Brandon's Equation

Teacher Notes

Math Milestones™

Central math concepts

One important use of the distributive property is as an aid for calculating a product that involves sums. For elementary-grades students, the first such problems arise when calculating products of multi-digit numbers; this is because every multi-digit number is a sum of terms. For example, 19 is a sum 10 + 9, which means we could calculate 5 × 19 mentally by thinking, "It's 50 + 45."

Upper-elementary students next apply the distributive property to products and sums that involve fractions. That development or progress is important because in middle grades, students will apply the distributive property to products and sums that involve not only whole numbers and fractions, but also rational numbers, real numbers, variables, and variable expressions. High school students will apply the distributive property not only to real numbers and variable expressions, but also to complex numbers, and perhaps to matrices. If the inner structure of arithmetic and algebra could be likened to a skeleton, then the distributive property would be the backbone.

In the symbol 378, there's nothing visible that tells us it refers to a sum; primary-grades students must learn to unpack 378 as 300 + 70 + 8. Similarly, upper-elementary students learn to unpack a symbol like 7.3 as $7 + \frac{3}{10}$. Mixed numbers are also sums, although this is not indicated by a symbol like $4\frac{1}{2}$. Fortunately, the symbol $4\frac{1}{2}$ is read aloud as "Four *and* one-half," where the word *and* supports correct interpretation of the symbol. Still, even high school students can sometimes accidentally misinterpret a symbol like $4\frac{1}{2}$ or have difficulty entering the number into a calculator. Their fluency in the conventions of algebra (in particular, the use of juxtaposition to indicate multiplication) can occasionally mislead them into thinking that $4\frac{1}{2}$ refers to the product $4 \times \frac{1}{2}$ rather than the sum $4 + \frac{1}{2}$.

A student who efficiently calculates $\frac{3}{4} \times 4\frac{1}{2} = \frac{3}{4} \times \frac{9}{2} = \frac{27}{8} = 3\frac{3}{8}$ has demonstrated a valuable fluency. Efficiently evaluating fraction products is important. Also important, especially as preparation for algebra, is being able to study expressions like the ones on either side of the equal sign in Brandon's equation, and *delay* evaluation of them in order to analyze the expression as an object with structure.

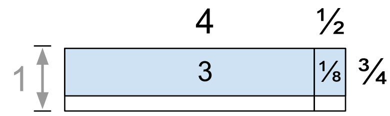
Thus, the most insightful explanation of the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$ arguably isn't simply to observe that the numerical values on either side of the equal sign are equal. For example, the same area models that illustrate the distributive property for whole numbers could be used to make sense of the numbers in Brandon's equation as partial products:

5:14 Brandon was reading his math book. He saw the equation $\frac{3}{4} \times (4 + \frac{1}{2}) = 3 + \frac{3}{8}$. He said, "I don't get it—where did the 3 and the $\frac{3}{8}$ come from?" Write an explanation that could answer Brandon's question.

Answer

Answers will vary. One kind of explanation involves calculating $\frac{3}{4} \times 4\frac{1}{2} = \frac{3}{4} \times \frac{9}{2} = \frac{27}{8} = 3\frac{3}{8}$, then recognizing that $3\frac{3}{8} = 3 + \frac{3}{8}$. A second kind of explanation involves an application of the distributive property. Because $\frac{3}{4} \times 4 = 3$ and $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$, we can multiply $\frac{3}{4} \times (4 + \frac{1}{2})$ by adding the products: $\frac{3}{4} \times (4 + \frac{1}{2}) = \frac{3}{4} \times 4 + \frac{3}{4} \times \frac{1}{2} = 3 + \frac{3}{8}$. This reveals 3 and $\frac{3}{8}$ as partial products. A third kind of explanation may take advantage of the bidirectionality of the equal sign by beginning the reasoning with $3 + \frac{3}{8}$ and working from right to left. (For example, $3 + \frac{3}{8}$ equals $3 \times (1 + \frac{1}{8})$, and $1 + \frac{1}{8}$ is a quarter of $4 + \frac{1}{2}$.) Answers may include such explanatory techniques as, for example: showing a math diagram, such as an area model; creating a simple word problem that makes sense of the numbers in Brandon's equation; and/or writing expressions and equations.

Click here for a student-facing version of the task.



To be sure, it is mathematically valid to explain Brandon's equation by observing that the numerical values on either side of the equal sign are equal—it's an ironclad proof. That explanation also demonstrates an understanding of the meaning of the equal sign. Thus, the purpose of task 5:14 isn't to differentiate between the students who explain things one way and the students who explain things the other way; the purpose is to bring different explanations together, and relate those explanations to one another, so that all students deepen their understanding and build a foundation for future learning.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: multiplying a whole number n by a fraction $\frac{m}{n}$; multiplying $\frac{1}{2}$ by a fraction; and basing multiplicative reasoning and distributive property reasoning on math diagrams.

Extending the task

How might students drive the conversation further?

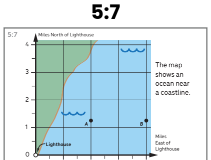
- Students could relate (or be asked to relate) Brandon's equation to familiar calculations involving whole numbers, such as 3 × 45 = 3 × 40 + 3 × 5, in which calculating a product involves adding terms.
- Similarly, students could think of, or be asked to think of, cases such as 7 × 99 = 700 - 7 in which calculating a product involves subtracting terms; or cases such as 6.4 ÷ 2, in which calculating a quotient involves adding or subtracting terms.

Related Math Milestones tasks

5:1

5:1 A school needed 240 four-packs of juice boxes for a field trip. However, the school accidentally bought 240 *six-packs* of juice boxes. How many extra juice boxes did the school buy?

5:7



Shipwrecks are at locations $A\ (2, 1\frac{1}{4})$ and $B\ (4, 1\frac{1}{4})$. Shipwrecks are also at locations $C\ (4, 3\frac{1}{2})$ and $D\ (2, 3\frac{1}{2})$. **(1)** Mark C and D on the map and shade rectangle $ABCD$. **(2)** Some believe there is sunken treasure in the region you shaded. How large is that region in mi^2?

Task **5:1 Juice Box Mixup** involves a whole-number calculation in context that can be considerably simplified by applying the distributive property. Task **5:7 Shipwrecks** involves calculating a product involving mixed numbers in context.

Refer to the Standards

5.NF.B.4a; MP.3, MP.5, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

Brandon's question specifically asks, "Where did the 3 and the $\frac{3}{8}$ come from?"—not just "I don't get it," or "Is this equation true?" The phrasing is intended to invite sense-making about the equation. Implicit in Brandon's question, perhaps, is that the confusing numbers 3 and $\frac{3}{8}$ are addends, so part of Brandon's confusion might be that a calculation like $\frac{3}{4} \times (4 + \frac{1}{2})$, in which we were supposed to multiply, got turned into an addition problem.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 5:14? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 5:14? In what specific ways do they differ from 5:14?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 5:14 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

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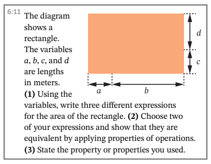
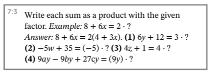
6:11



6:11 The diagram shows a rectangle. The variables a, b, c, and d are lengths in meters. **(1)** Using the variables, write three different expressions for the area of the rectangle. **(2)** Choose two of your expressions and show that they are equivalent by applying properties of operations. **(3)** State the property or properties you used.

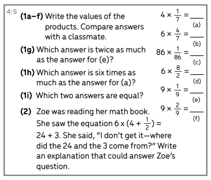
7:3



7:3 Write each sum as a product with the given factor. *Example:* $8 + 6x = 2 \cdot ?$
Answer: $8 + 6x = 2(4 + 3x)$. **(1)** $6y + 12 = 3 \cdot ?$
(2) $-5w + 35 = (-5) \cdot ?$ **(3)** $4z + 1 = 4 \cdot ?$
(4) $9ay - 9by + 27cy = (9y) \cdot ?$

In later grades, the distributive property will become the central principle in rewriting expressions; see tasks **6:11 Area Expressions** and **7:3 Writing Sums as Products**.

4:5



4:5 **(1a–f)** Write the values of the products. Compare answers with a classmate.

(1g) Which answer is twice as much as the answer for (e)?

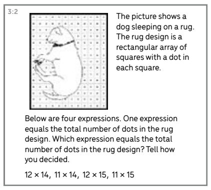
(1h) Which answer is six times as much as the answer for (a)?

(1i) Which two answers are equal?

(2) Zoe was reading her math book. She saw the equation $6 \times (4 + \frac{1}{2}) = 24 + 3$. She said, "I don't get it—where did the 24 and the 3 come from?" Write an explanation that could answer Zoe's question.

$4 \times \frac{1}{7} = \underline{\quad}$ (a)
$6 \times \frac{4}{7} = \underline{\quad}$ (b)
$86 \times \frac{1}{86} = \underline{\quad}$ (c)
$6 \times \frac{8}{2} = \underline{\quad}$ (d)
$9 \times \frac{1}{9} = \underline{\quad}$ (e)
$9 \times \frac{2}{9} = \underline{\quad}$ (f)

3:2

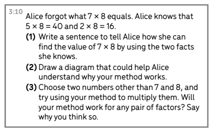


3:2 The picture shows a dog sleeping on a rug. The rug design is a rectangular array of squares with a dot in each square.

Below are four expressions. One expression equals the total number of dots in the rug design. Which expression equals the total number of dots in the rug design? Tell how you decided.

$12 \times 14,\ 11 \times 14,\ 12 \times 15,\ 11 \times 15$

3:10



3:10 Alice forgot what 7×8 equals. Alice knows that $5 \times 8 = 40$ and $2 \times 8 = 16$.

(1) Write a sentence to tell Alice how she can find the value of 7×8 by using the two facts she knows.

(2) Draw a diagram that could help Alice understand why your method works.

(3) Choose two numbers other than 7 and 8, and try using your method to multiply them. Will your method work for any pair of factors? Say why you think so.

In earlier grades, task **4:5 Fraction Products and Properties** (part (2)) involves an analogue of Brandon's equation in which the multiplier of $(4 + \frac{1}{2})$ is a whole number (6) rather than a fraction ($\frac{3}{4}$). Task **3:2 Hidden Rug Design** emphasizes viewing multiplication expressions as objects with structure and meaning. Task **3:10 Alice's Multiplication Fact** involves applying the distributive property as part of the process of learning the multiplication table.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

5:14 Brandon's Equation

Teacher Notes

Math Milestones™

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?