7:4 "Foul Play"

Teacher Notes

Math Milestones™

Central math concepts

"Will I get rained on today?" and "Will I get struck by a meteorite today?" are two questions that could both be answered "It's possible," but thanks to probability we don't have to leave it at that. Probability allows us to rate the likelihood of two events on a 0–1 scale, allowing us to make quantitative comparisons using ratios and multiplicative comparison. For example, we might estimate that getting struck by a meteorite today is hundreds of times, or thousands of times, or millions of times less likely than getting rained on today. In this way probability makes it possible to be quantitative about likelihood, belief, and risk, giving us a powerful tool for coping with uncertainty.

In games and sports, strategy and decision making can be the difference between winning and losing. In task 7:4, a defender made a split-second decision and later wondered if it was best. Simulation is a good way to analyze such questions. For example, to simulate three successive free throws, a student could randomly generate three whole numbers between 1 and 10 using concrete objects or technology. (As one technology option, the figure shows a Google die rolling feature with a ten-sided die option; a non-pictorial version is here).

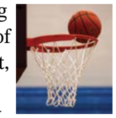
Ahead of time, designate a particular number, such as 4, to represent a missed free throw. The reason one in ten numbers is designated as a "miss" is that the 9 in 10 chance the Pistons player makes a free throw corresponds to a 1 in 10 chance the Pistons player misses a free throw.

Example: Designate 4 as representing a missed free throw and suppose the random numbers generated are 7, 4, and 3. Then this sequence represents the compound event of "make the first free throw, miss the second, and make the third." That sequence doesn't win the game for the Pistons as time is running out, because the score had been 100-98, so the Pistons player must make all three free throws to win the game as time is running out.

A dozen students or a dozen pairs of students could generate a sequence of three random numbers, with each student/pair recording whether or not the sequence corresponds to a win for the Pistons as time is running out. The table collects the results from a dozen hypothetical students/pairs.

Student/pair	Random number sequence	Interpretation	Pistons win as time is running out?
A	7, **4**, 3	make, **miss**, make	No
B	7, 10, 9	make, make, make	Yes
C	3, 5, 6	make, make, make	Yes

7:4 *"Foul Play."* The Hawks were leading the Pistons in basketball by a score of 100-98. Just as time was running out, a Pistons player tried a 3-point shot. His defender had two choices: allow the shot, or stop it by fouling the Pistons player. Fouling would give the Pistons player 3 one-point free throws. The defender chose to foul and later wondered if it was a good choice. **(1)** To analyze the defender's choice, let's assume that for the Pistons player, every 3-point shot has probability $\frac{1}{3}$ of going in, and every free throw has probability 90% of going in. **(a)** If the defender allows the shot, what is the probability that the shot wins the game as time runs out? **(b)** If the defender stops the shot by fouling, estimate the probability that the free throws win the game. **(2)** Write a paragraph arguing for or against the defender's choice, based on probability calculations and/or simulations.



Answer

(1) (a) An exact answer of $\frac{1}{3}$, $0.\overline{3}$, or $33.\overline{3}$%, or a truncated base-ten form such as 0.333, 33%, etc. **(b)** The probability is about 73%; any estimate from about 53% to about 93% should be considered accurate, since these values are all significantly greater than 33%. **(2)** Responses may vary but should at minimum acknowledge that, given the assumptions of the model, the probability of the Pistons player making 3 free throws in a row is significantly greater than the probability of the Pistons player making the 3-point shot. Answers could also include an evaluation of the reasonableness of the assumptions of the model, and/or a second analysis based on assumptions felt to be more reasonable.

Note: It is possible that the Pistons player could shoot free throws and make exactly two, sending the game to overtime—but the task doesn't require students to address that possibility in their response. Students could still choose to address that possibility however, either qualitatively or quantitatively.

Click here for a student-facing version of the task.

Student/pair	Random number sequence	Interpretation	Pistons win as time is running out?
D	7, 9, 3	make, make, make	Yes
E	6, 10, **4**	make, make, **miss**	No
F	9, 3, 8	make, make, make	Yes
G	9, 5, 8	make, make, make	Yes
H	9, 1, 6	make, make, make	Yes
I	3, **4**, 1	make, **miss**, make	No
J	7, 9, 2	make, make, make	Yes
K	5, 2, 8	make, make, make	Yes
L	**4**, 1, 2	**miss**, make, make	No

In this simulation, it happened 8 times that the Pistons won on free throws as time was running out, and it happened 4 times that they did not. Based on this simulation, an estimate of the probability that the free throws win the game as time is running out is $\frac{8}{12}$ or about 67%.† The estimate of 67% is a reasonable answer for part (1b) of the task.

It can be tempting to see meaningless patterns in random numbers, such as by observing that the table contains many more odd numbers than even numbers. This is one form of common probability misconception (see "Clustering Illusion" and related topics). Reflective experiences with chance processes can build intuition about signal and noise, and can surface widespread misconceptions about chance and variability that interfere with sound decision making.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: converting between fractions, decimals, and percents and using number sense of fractions, decimals, and percents; comparing numbers multiplicatively; and using technology.

Extending the task

How might students drive the conversation further?

- Students could replace the probabilities $\frac{1}{3}$ and 90% with other values, to investigate the situation for a more accurate three-point shooter or a less accurate free throw shooter. For example, a student knowledgeable about NBA basketball might observe that most players who shoot 90% from the free throw line tend to shoot significantly better than 33% from the three-point line. Note that a simulation with more typical values of these numbers might require more simulation data to accurately estimate the probabilities in the task.
- Students might discuss what a good strategy could be if the Pistons player were very accurate at both three-point shots *and* free throws.

Refer to the Standards

7.SP.C; MP.4, MP.5. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

- Students may not have knowledge of basketball rules, and/or students may not be familiar with typical game situations. Knowledge of rules could be shared, and the task could be acted out to illustrate the situation. Another benefit of acting out the situation is that it could remove the need for students to read the text of the problem. Key information could be elicited and summarized as bullet points.
- The task includes an explicit modeling assumption that "for the Pistons player, … every free throw has probability 90% of going in." The stated assumption is a strong one; for example, the model excludes consideration of whether the Pistons player will get nervous on the free throw line.
- The defender *later* wondered if fouling had been a good choice. The task doesn't suggest that a real basketball player would, or should, perform a quantitative probability analysis in the split-second game situation.

Related Math Milestones tasks

7:11

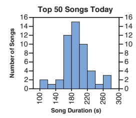
7:11 Nechama is shopping online for a ticket to a play. Website A offers a discount of $7.50 off the theater price. Website B offers a discount of 25% off the theater price. **(1)** Is it mathematically possible that Website A is a better deal than Website B? **(2)** Is it mathematically possible that Website B is a better deal than Website A? *Prove your answers.*

6:10

6:10 In the month of February 2021, there were 20 weekdays and 8 weekend days. Here are some questions about that month. **(1)** (Circle all of the correct answers.) The ratio of weekdays to weekend days was 20:8 10:4 5:2 5:7. **(2)** There were ___ times as many weekdays as weekend days. **(3)** True or false: $\frac{5}{7}$ of the days that month were weekdays. **(4)** Approximately what percent of the days that month were weekdays?

6:7

6:7 **(1)** Look up the 50 top songs on a music streaming service. Type each song's duration into a spreadsheet. **(2)** Write a sentence about the data giving a measure of center and a measure of variability. **(3)** Make a histogram of the data.* **(4)** Write a sentence describing the overall pattern of the distribution and any striking deviations from the overall pattern. **(5)** Imagine that one year from now, you go back online and repeat (1)–(4). In what ways would you expect the data distribution to look similar? What differences would you expect to see?



*Use this histogram for (4) and (5) if you don't do (3).

Task **7:11 Ticket Offers** involves decimals and percents as well as the role of the quantitative in decision making.

In later grades, high school students will develop techniques in probability such as multiplying the probabilities of independent events to calculate the probability of a compound event.

In earlier grades, task **6:10 Weekdays and Weekend Days** relates ratios to fractions and percents. Task **6:7 Song Length Distribution** involves a data distribution that arises from sampling variability, as compared to the chance variability which is the issue in task 7:4.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 7:4? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 7:4? In what specific ways do they differ from 7:4?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 7:4 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† The table was produced using a random number generator. Note that with twelve or more students/pairs, it is very likely that the simulation will result in an estimate in the range from about 53% to 93%. A greater number of three-number sequences could be generated to improve the estimate.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?