7:9 Calculating with Rational Numbers

Teacher Notes

Central math concepts

The properties of operations allow calculation with positive fractions to be extended to calculation with signed rational numbers.

$(a + b) + c = a + (b + c)$

$a + b = b + a$

$a + 0 = 0 + a = a$

For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$.

$(a \times b) \times c = a \times (b \times c)$

$a \times b = b \times a$

$a \times 1 = 1 \times a = a$

For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$.

$a \times (b + c) = a \times b + a \times c$

On the one hand, many aspects of calculating with rational numbers are already present when calculating with whole numbers or positive fractions. For example, the additive identity property of 0 is used already in kindergarten with whole numbers, and 0 has the same property in the rational number system. As another example, given any positive fraction, there exists a positive fraction (namely the reciprocal) whose product with the given fraction is 1; and reciprocals (multiplicative inverses) exist for all nonzero rational numbers too. Some other aspects of earlier-grades work with number and operations that will continue in the rational number system include:

- Associativity of addition and multiplication.
- The multiplicative identity property of 1.
- Commutativity of addition and multiplication.
- The relationship between addition and subtraction: $C - A$ is the unknown addend in $A + \square = C$.
- The relationship between multiplication and division: $C \div A$ (with A nonzero) is the unknown factor in $A \cdot \square = C$.
- The relationship between multiplication and addition: $A \cdot (B + C) = A \cdot B + A \cdot C$.

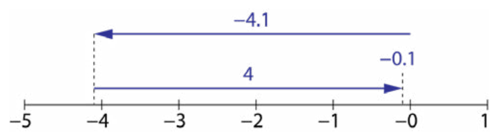
One new aspect of the rational number system is the existence of additive inverses. There is no positive number that solves the equation $1 + x = 0$, but there is one rational number that solves the equation, namely $x = -1$. More generally, every rational number A has a unique additive inverse $-A$ with the defining property $A + -A = 0$.

Also new in the rational number system is the question of how to evaluate products like $-1 \cdot 5$ or $-3 \cdot -4$. In principle, the properties also answer these questions. For example, to evaluate the product $-1\cdot5$, one could add 5 to the product and see what happens: $5 + -1\cdot5 = 1\cdot5 + -1\cdot5 = (1 + -1)\cdot5 = 0\cdot5 = 0$. Comparing the first and last expressions, we have $5 + -1\cdot5 = 0$, which says that $-1\cdot5$ is the additive inverse of 5, or -5. Generalizing such observations justifies procedural guidance such as "negative times positive equals negative."

7:9 **(1)** Calculate. **(a)** $-4.1 + 4$ **(b)** $5 \div (-6)$ **(c)** $-1(-1 - 1)$ **(d)** $2 - (-\frac{1}{2})$ **(e)** $(-\frac{3}{8})(-8)$ **(f)** $0 - \frac{1}{3}$ **(g)** $\frac{1}{7.9} * 7.9$ **(h)** $(\frac{1}{2} - \frac{1}{4})(-9 + 9)$.
(2) Show calculation 1(a) on a number line.

Answer

(1) (a) -0.1. **(b)** $-\frac{5}{6}$. **(c)** 2. **(d)** $2\frac{1}{2}$ or $\frac{5}{2}$. **(e)** 3. **(f)** $-\frac{1}{3}$. **(g)** 1. **(h)** 0. **(2)** Answers may vary but should reveal or support an explanation of how the result comes about. See the figure for an example.



Click here for a student-facing version of the task.

Refer to the Standards

7.NS.A; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Procedural skill and fluency

Additional notes on the design of the task

- The varied calculations in part (1) seek to portray positive and negative fractions and decimals as an integrated system of numbers that can be operated on and transformed.

The prominence of the properties of operations in calculating with rational numbers means that calculating with rational numbers is more like doing algebra than was the case when students calculated with multi-digit whole numbers. There isn't a standard algorithm for evaluating expressions like $-1(-1 - 1)$ or $(\frac{1}{2} - \frac{1}{4})(-9 + 9)$. Instead, there are choices to make, such as whether to evaluate $\frac{1}{2} - \frac{1}{4}$ first or evaluate $-9 + 9$ first. Those choices require comprehension of the structure of expressions, as well as fluency with the syntax of expressions and their conventions (such as omitting the multiplication sign or using the same symbol for subtraction as for negation). In some ways, calculating with rational numbers resembles opportunistic mental calculations from earlier grades, as in problems like 4,999 + 12, in which properties of operations and relationships between operations can be used to rewrite the given expression in a more convenient form, in this case 4,999 + 1 + 11 = 5,000 + 11. Calculation in the elementary grades was never only algorithmic[†], and in the middle grades and high school it seldom ever is.[‡]

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: understanding quotients of whole numbers as fractions; understanding mixed numbers as sums of whole numbers and fractions; multiplying a whole number by a fraction; relating fractions and decimals; using grouping symbols; and looking for structure in expressions.

Extending the task

How might students drive the conversation further?

- Students could generate and compare several possible early steps for a given calculation, such as $(-\frac{3}{8})(-8) = (\frac{3}{8})(8) = (\frac{3}{8})(\frac{8}{1})$ vs. $(-\frac{3}{8})(-8) = (\frac{3}{8})(8) = 3 \cdot \frac{1}{8} \cdot 8$ or other possibilities.
- In discussing errors, students could suggest ways to rewrite the expression so its structure is more transparent, for example:

$$-1(-1 - 1) = (-1)(-1 + -1).$$

Related Math Milestones tasks

7:12

7:12 In 1972 in Loma, Montana, the temperature changed from $-54°F$ to $+49°F$ in a 24-hr period. Calculate the average rate at which the temperature changed. Answer to the nearest tenth in units of degrees/hr.

7:3

7:3 Write each sum as a product with the given factor. *Example:* $8 + 6x = 2 \cdot ?$ *Answer:* $8 + 6x = 2(4 + 3x)$. **(1)** $6y + 12 = 3 \cdot ?$ **(2)** $-5w + 35 = (-5) \cdot ?$ **(3)** $4z + 1 = 4 \cdot ?$ **(4)** $9ay - 9by + 27cy = (9y) \cdot ?$

7:5

7:5 *Pencils down* Think about the equation $x + 4\frac{1}{8} = \frac{2}{3}$. Is there a positive number that solves it? Is there a negative number that solves it? Tell how you decided.

Task **7:12 Temperature Change** involves arithmetic with signed rational numbers in context, and task **7:3 Writing Sums as Products** involves rewriting algebraic expressions with rational number coefficients. Task **7:5 Is There a Solution? (Addition)** focuses on an equation that has no solution in the positive numbers but that can be solved in the rational number system.

Curriculum connection

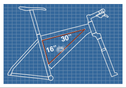
1. In which unit of your curriculum would you expect to find tasks like 7:9? Locate 2–3 similar tasks in that unit. How are the tasks similar to each other, and to 7:9? In what specific ways do they differ from 7:9?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 7:9 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

8:6

8:6 Write as a fraction in lowest terms: **(1)** $1.041\overline{6}$. **(2)** $3^2 \cdot 3^{-5}$.

8:3

8:3 On this blueprint for building a bike, part of the bike is shaped like a right triangle. The longest side length is illegible because water spilled on the blueprint. Calculate that side length.



In later grades, task **8:6 Rational Form** involves connections between the rational number system and repeating decimals. Task **8:3 Bicycle Blueprint** involves the Pythagorean theorem, the full meaning of which extends the concept of number beyond rational numbers.

6:5

6:5 **(1)** Which of the numbers $5, -7, \frac{2}{3}, -\frac{1}{2}$ is farthest from 0 on a number line? Which is closest to 0? **(2)** True or False: $\frac{1}{2} > -8$. **(3)** Explain why $-(-0.2) = 0.2$ makes sense.

6:13

6:13 w Think about the **equation** $241p = \frac{3}{4}$. Is there a whole number that solves it? Is there a non-whole number that solves it? Convince a classmate that your answers are right.

6:14

6:14 *Pencil and paper* **(1)** $81.53 \div 3.1 = ?$ **(2)** $\frac{7}{8} \div \frac{2}{3} = ?$ **(3)** Check both of your answers by multiplying.

In earlier grades, task **6:5 Positive and Negative Numbers** concentrates on pre-arithmetic properties of signed rational numbers and their locations on the number line. Task **6:13 Is There a Solution? (Multiplication)** focuses on an equation that has no solution in the whole numbers but that can be solved in the positive fractions. Task **6:14 Dividing Decimals** and Fractions is a procedural task involving quotients of fractions and decimals.

† Adding It Up (2001), p. 121.

‡ William McCallum (2008), "Mindful Manipulation: What Algebra Do Students Need for Calculus?" (presentation)

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

