8:4 System Solutions

Teacher Notes

Central math concepts

A two-variable equation states a condition which the values of the two variables must satisfy. For example, the equation $x + y = 0$ states a condition on x and y that is satisfied for certain values of x and y (such as $x = 5$ and $y = -5$) but not satisfied for other values of x and y (such as $x = 1$ and $y = 1$). A solution to a two-variable equation is an ordered pair of values—one value for each variable—that makes the equation true when those values are substituted into the equation for the variables.

A two-variable equation for real numbers x and y is called linear when it can be put in the form $Ax + By = C$, where A, B, and C are specific real numbers with A and B not both zero. For example, $2x + 3y = 12$ is a linear equation. The reason for the name "linear equation" is that if all the solutions of the equation (that is, all the ordered pairs of real numbers that satisfy the equation) are plotted as points in the xy coordinate plane, then the resulting set of points is a line. This can be seen by rewriting the equation $2x + 3y = 12$ first in the form $y = -\frac{1}{6}x + 4$, then rewriting it again in the form $\frac{(y - 4)}{(x - 0)} = -\frac{1}{6}$. This last equation is the statement that the slope calculated using the particular point (0, 4) and any other point on the graph is always the same. The condition of constant slope is equivalent to the graph being a straight line.†

Since a two-variable equation states a condition which the values of the two variables must satisfy, a system of two simultaneous equations states two conditions that the values of the variables must satisfy simultaneously. In the case of two linear equations, the graph of the solutions for each equation is a straight line; a point belongs to a given line if and only if the coordinates of the point solve the corresponding equation. A point belongs to both lines simultaneously if and only if the coordinates of the point solve both equations simultaneously. This gives rise to the strategy of solving a system of two simultaneous linear equations in two variables by graphing the solutions of each equation and looking for intersections of the graphs. If an intersection point is found, then the approximate coordinates of the intersection point can be taken for an approximate solution to the system, or the coordinates could be substituted for the variables in both equations to see if they are an exact solution.

Two lines can intersect in 1, 0, or infinitely many points, and a system of two simultaneous linear equations can have 1, 0, or infinitely many solutions. A key factor in the nature of the solutions of a system of two simultaneous linear equations in two variables is whether the corresponding lines are parallel. Non-vertical lines are parallel when their slopes are equal, which can be seen directly from the corresponding equations if they are in $y = mx + b$ form: the lines are parallel when the m coefficient is the same in the two equations. If the equations are in the form $Ax + By = C$, then one can show that the corresponding non-vertical lines are parallel when the ratio A:B is the same for the two equations.

8:4 **(1)** Decide whether each system has exactly one solution, infinitely many solutions, or no solutions. **(2)** For one system, justify your decision to your classmates in two ways: **(a)** drawing graphs of solutions; **(b)** algebraically.

$$\begin{Bmatrix} y = \frac{2}{3}x + 1 \\ y = \frac{2}{3}x + 2 \end{Bmatrix} \quad \begin{Bmatrix} d = 100 - 4t \\ d = 3.5 + t \end{Bmatrix} \quad \begin{Bmatrix} \frac{1}{8}Q + \frac{3}{8}R = -1 \\ Q + 3R = -8 \end{Bmatrix}$$

Answer

(1) The left-hand system has no solutions. The middle system has exactly one solution. The right-hand system has infinitely many solutions. **(2) (a)** Answers may vary depending on which system is chosen. **(b)** Answers may vary depending on which system is chosen and what algebraic arguments are given to justify the decision.

An algebraic justification that the left-hand system has no solutions could involve: (i) substituting, say, $\frac{2}{3}x + 2$ into the first equation for y, and deducing an absurd conclusion; (ii) observing that $y - \frac{2}{3}x$ cannot equal 2 if it equals 1; or another appropriate algebraic argument (see also argument (ii) in the next paragraph).

An algebraic justification that the middle solution has exactly one solution could involve: (i) substituting, say, $3.5 + t$ into the first equation for d, and deducing that $t = 19.3$, $d = 22.8$ is the only possible solution—then checking that $t = 19.3$, $d = 22.8$ does indeed solve the equation; (ii) observing that the right-hand sides of the two equations could define two functions of time, one function that decreases without limit from a large initial value of 100, and another that increases without limit from a small initial value of 3.5—in which case, there is necessarily a moment of time when the value increasing from 3.5 must equal the value decreasing from 100; or another appropriate algebraic argument.

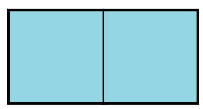
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: using properties of operations to rewrite expressions; analyzing and solving one-variable equations; relating a two-variable equation to the graph of its solutions; and solving equations as a process of reasoning.

Extending the task

How might students drive the conversation further?

- Students could relate the nature of solutions of systems of simultaneous linear equations in two variables to the nature of solutions of single linear equations in one variable by considering the following sequence of word problems:
  - Find the side length of a square if the perimeter is 6 units greater than the side length. (A constraint equation for the problem might be $4x = 6 + x$. This equation could result from function thinking: For a square, the perimeter p is a function of the side length, $p = 4x$. A second function could be defined as $q = x + 6$. For what value of x does $p = q$?)
  - Find the side length of a square if two copies of the square when joined form a rectangle with perimeter equal to six times the side length of the square. (An equation model for the problem might be $x + x + x + x + x + x = 6x$.)

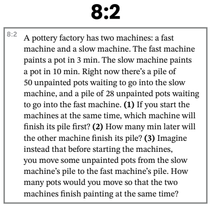


  - Find the side length of a square if increasing the side length by 1 unit causes the perimeter to increase by 2 units. (An equation model for the problem might be $4(x + 1) = 4x + 2$.)

  Which problem has 1 solution? 0 solutions? Infinitely many solutions?

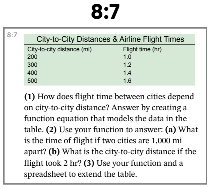
Related Math Milestones tasks

8:2



8:2 A pottery factory has two machines: a fast machine and a slow machine. The fast machine paints a pot in 3 min. The slow machine paints a pot in 10 min. Right now there's a pile of 50 unpainted pots waiting to go into the slow machine, and a pile of 28 unpainted pots waiting to go into the fast machine. **(1)** If you start the machines at the same time, which machine will finish its pile first? **(2)** How many min later will the other machine finish its pile? **(3)** Imagine instead that before starting the machines, you move some unpainted pots from the slow machine's pile to the fast machine's pile. How many pots would you move so that the two machines finish painting at the same time?

8:7



8:7

City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

Part (3) of **8:2 Pottery Factory** states a condition that could be expressed as a single-variable equation or as a pair of simultaneous equations analogous in form to the left-hand system or middle system in task 8:4. Part (2b) of **8:7 Flight Times and Distances** involves solving a constraint equation arising from a stated condition.

Answer (continued)

An algebraic justification that the right-hand system has infinitely many solutions could involve: (i) multiplying the first equation through by 8 to show that the two equations are the same (and noting that a single linear equation in two variables has infinitely many solutions); (ii) solving both equations for Q and finding $Q = -8 - 3R$ in both cases, so that all the system requires is a free choice of value for R followed by a calculation of the required corresponding value of Q; or another appropriate algebraic argument.

Click here for a student-facing version of the task.

Refer to the Standards

8.EE.C.8; MP.5, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Procedural skill and fluency

Additional notes on the design of the task

- Both equations in the left-hand system have "$y = mx + b$" form, which may invite a graphing approach for students practiced in translating between graphs and equations in this form.
- Both equations in the middle system have a "Quantity = initial value + (rate)·(time)" form, which may invite a functional interpretation of the algebra.

7:13

7:13 A 15.1-in long wire is bent into the shape of a circle with 2.9 in left over. To the nearest 0.1 in, what is the diameter of the circle?

7:14

7:14 Rose and Liba both solved this problem: *Jannat has 4 packs of balloons and 5 single balloons—29 balloons in all. How many balloons are in a pack?* Explain both of Rose's steps. Check that Liba's equations are all true statements about the balloons.

Rose	**Liba**
$29 - 5 = 24$	Let x be the # of balloons in a pack.
$24 \div 4 = 6$	$4x + 5 = 29$
	$4x = 24$
	$x = 6$

7:8

7:8 In 2018, an oil company rented an oil rig for \$100,000 per day. The company drilled a well and started pumping oil. **(1)** How much oil must be sold each day to equal the rental cost? Note: 42 gal of oil could be sold for \$70 in 2018. **(2)** The company estimates that the profit, P, in millions of dollars, after pumping oil for D days is $P = 0.5D - 40$. **(a)** What is the profit after the first day of pumping oil? **(b)** Make a table of pairs of values (D, P) and graph the ordered pairs. **(c)** How can the company make \$30M of profit? **(3)** An equivalent expression for P is $0.5(D - 80)$. How does the 80 in this expression relate to the company's situation?

6:13

6:13 *Pencils down* Think about the equation $241p = \frac{3}{4}$. Is there a whole number that solves it? Is there a non-whole number that solves it? Convince a classmate that your answers are right.

7:5

7:5 *Pencils down* Think about the equation $x + 4\frac{1}{8} = \frac{2}{3}$. Is there a positive number that solves it? Is there a negative number that solves it? Tell how you decided.

In earlier grades, **7:13 Wire Circle**, **7:14 Comparing Rose and Liba's Solutions**, and part (2c) of **7:8 Oil Business** involve solving a constraint equation arising from a stated condition. Tasks **6:13 Is There a Solution? (Multiplication)** and **7:5 Is There a Solution? (Addition)** emphasize the idea that constraint equations can be interpreted as questions.

Additional notes on the design of the task (continued)

- The equations in the right-hand system are intended to have conspicuous numbers 3 and 8 so as to prompt looking for and making use of structure.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:4? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 8:4? In what specific ways do they differ from 8:4?

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:4 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† In the case $A = 0$, the linear equation $Ax + By = C$ is equivalent to the equation $y = C/B$, the solutions of which graph as a horizontal line; in the case $B = 0$, then the linear equation $Ax + By = C$ is equivalent to the equation $x = C/A$, the solutions of which graph as a vertical line.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

