8:1 Xavier's Notes

Teacher Notes

Central math concepts

In a situation with two quantities, "assuming a linear model" means assuming that changes in one quantity are directly proportional to changes in the other quantity (with a fixed constant of proportionality). Said another way, assuming a linear model means assuming that the rate of change of one quantity with respect to another quantity remains constant. Graphically, if we were to imagine graphing ordered pairs for the relationship, then assuming a linear model means assuming that the slope would be the same when calculated using any two chosen points.

In this task specifically, one can calculate Xavier's average rate for summarizing 5 pages between 4:45 pm to 6:00 pm; a linear model would then imply that Xavier will summarize the last 7 pages at the same average rate.

Assuming a linear model can be a useful way to analyze a situation, even if this assumption is only an approximation to the reality. In fact, Xavier's rate of summarizing is unlikely to be mathematically constant; probably there are some pages that can be summarized quickly, while others can only be summarized more slowly. Or Xavier may become impatient towards evening and therefore begin summarizing at a faster rate. One doesn't have to *believe* a linear model in order to find out *what would follow from it*.

Modeling always involves a judgment about when simplifying a situation will or won't yield useful insights. In Xavier's case, assuming a linear model might be useful. For example, using a linear model at 6:00 pm might allow Xavier to draw the conclusion that 1 hour and 45 minutes is going to be too long to continue working on the assignment, and therefore he might purposefully pick up the pace.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: distinguishing between clock readings and elapsed time; calculating a unit rate; finding an unknown quantity in a proportional relationship; graphing ordered pairs; and calculating the slope of a line given two points.

Extending the task

How might students drive the conversation further?

- Students could discuss the possible usefulness of assuming a linear model in this situation, and also detail the simplifications that are being made by that assumption.
- Students could relate task 8:1 to scatter plots, because if we had access to more data about Xavier's progress, we could create a

8:1 Xavier's assignment for science class was to write notes to summarize a chapter in his textbook. At 4:45 p.m., he had 12 pages left to summarize. At 6:00 p.m., he had 7 pages left. Assuming a linear model, about how many more hours will it take him to finish summarizing?

Answer

Assuming a linear model, it will take Xavier about 1 hour and 45 minutes longer to finish summarizing. Any answer from 1.5 hours to 2 hours is reasonable. Answers may be expressed in hours, minutes, or hours and minutes.

Click here for a student-facing version of the task.

Refer to the Standards

8.F.B.4; MP.1, MP.2, MP.4, MP.5, MP.6. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

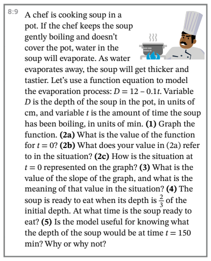
Additional notes on the design of the task

- Because students may not be familiar with problems that ask them to assume a linear model, it is fine to explain to the students what it means to "assume a linear model," using terms that will best make sense to the students.

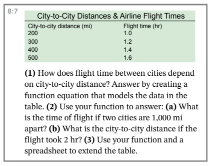
scatter plot showing more pairs of values (with one value in each pair being elapsed time, and the other value being the number of pages remaining). If the scatter plot did not suggest a linear association, then we might reevaluate the choice of a linear model.

Related Math Milestones tasks

8:9


8:9 A chef is cooking soup in a pot. If the chef keeps the soup gently boiling and doesn't cover the pot, water in the soup will evaporate. As water evaporates away, the soup will get thicker and tastier. Let's use a function equation to model the evaporation process: $D = 12 - 0.1t$. Variable D is the depth of the soup in the pot, in units of cm, and variable t is the amount of time the soup has been boiling, in units of min. **(1)** Graph the function. **(2a)** What is the value of the function for $t = 0$? **(2b)** What does your value in (2a) refer to in the situation? **(2c)** How is the situation at $t = 0$ represented on the graph? **(3)** What is the value of the slope of the graph, and what is the meaning of that value in the situation? **(4)** The soup is ready to eat when its depth is $\frac{2}{3}$ of the initial depth. At what time is the soup ready to eat? **(5)** Is the model useful for knowing what the depth of the soup would be at time $t = 150$ min? Why or why not?

8:7

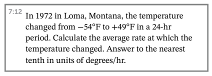

8:7

City-to-City Distances & Airline Flight Times

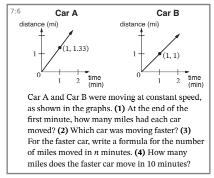
City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

7:12

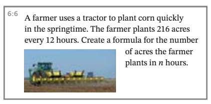

7:12 In 1972 in Loma, Montana, the temperature changed from −54°F to +49°F in a 24-hr period. Calculate the average rate at which the temperature changed. Answer to the nearest tenth in units of degrees/hr.

7:6

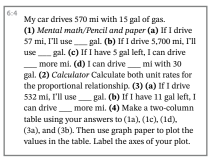


Car A and Car B were moving at constant speed, as shown in the graphs. **(1)** At the end of the first minute, how many miles had each car moved? **(2)** Which car was moving faster? **(3)** For the faster car, write a formula for the number of miles moved in n minutes. **(4)** How many miles does the faster car move in 10 minutes?

6:6


6:6 A farmer uses a tractor to plant corn quickly in the springtime. The farmer plants 216 acres every 12 hours. Create a formula for the number of acres the farmer plants in n hours.

6:4


6:4 My car drives 570 mi with 15 gal of gas. **(1)** *Mental math/Pencil and paper* **(a)** If I drive 57 mi, I'll use ___ gal. **(b)** If I drive 5,700 mi, I'll use ___ gal. **(c)** If I have 5 gal left, I can drive ___ more mi. **(d)** I can drive ___ mi with 30 gal. **(2)** *Calculator* Calculate both unit rates for the proportional relationship. **(3)** **(a)** If I drive 532 mi, I'll use ___ gal. **(b)** If I have 11 gal left, I can drive ___ more mi. **(4)** Make a two-column table using your answers to (1a), (1c), (1d), (3a), and (3b). Then use graph paper to plot the values in the table. Label the axes of your plot.

Other Math Milestones tasks for Grade 8 that prominently feature linear models or linear functions are **8:9 Water Evaporation Model** and **8:7 Flight Times and Distances**.

In earlier grades, task **7:12 Temperature Change** involves the concept of average rate. Tasks **7:6 Car A and Car B**, **6:6 Planting Corn**, and **6:4 Gas Mileage** prominently feature proportional relationships.

Additional notes on the design of the task (continued)

- The task doesn't require students to use equations, tables, or graphs to represent the linear function model. Students may choose to use any or all of those representations in solving or discussing the problem. Or they may choose to "keep the linear function in their head" and solve the problem by simply performing appropriate numerical calculations such as 12 - 7 = 5, 6 - 4.75 = 1.25, 5/1.25 = 4, 7/4 = 1.75. In that case, students could be asked to describe to a partner what quantities they are calculating (for example, 4 is Xavier's rate in pages per hour), and the partners could look for ways those quantities can be seen in other students' representations.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:1? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 8:1? In what specific ways do they differ from 8:1?

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:1 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

8:1 Xavier's Notes

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:2 Pottery Factory

Teacher Notes

Central math concepts

One kind of equation that is used in algebra is a *function equation* that presents the rule for a relationship between two covarying quantities in a situation. An example of a function equation might be $C = 250 + 10n$, where n is the number of cell phone minutes used in a month and C is the resulting monthly charge. (See **6:6 Planting Corn**, **7:6 Car A and Car B** part (3), **8:7 Flight Times and Distances**, and **8:9 Water Evaporation Model** for additional examples).

Another kind of equation is a *constraint equation*. A constraint equation states a condition that must be satisfied. A constraint equation can be viewed as asking a question: Which values from a specified set, if any, make the equation true? Solving a constraint equation is a process of reasoning resulting in a complete answer to that question. An example of a constraint equation might be $250 + 10n = 1000$. Does any positive value of n make this equation true, and if so, what are the value(s) of n that make the equation true?

Function equations and constraint equations differ in an important way. Whereas a constraint equation poses a question about what its solutions are, a function equation doesn't pose a question. Function equations aren't asking, they're telling: telling you the rule for how one quantity depends on another. The function equation $C = 250 + 10n$ expresses a rule for the value of C, given any n. By contrast, the constraint equation $250 + 10n = 1000$ is something like a puzzle: what value(s) of n make the equation true? Constraint equations invite you to unravel them, to root out the unknown value(s) of the quantity or quantities they determine yet conceal.

An important point of connection between function equations and constraint equations is that building both kinds of equations requires applying operations to a variable in order to build an expression. One calculates with the variable as if it were a number, applying the meanings and properties of operations. In the case of a function equation, the expression built up in this way defines the rule for the function. The expression $250 + 10n$ defines the rule for the monthly charge given an input number of minutes, n. Meanwhile, in the case of a constraint equation, it often happens that some quantity in the problem can be calculated by two different routes, producing two inequivalent expressions that must nevertheless have the same value. The statement that these two expressions have the same value then becomes a constraint equation for the problem.

For example, a condition might be stated as, "My monthly charge for December was \$100 less than my monthly charge for November because I sent half as many text messages in December compared to November." This rather intricate condition could be represented by the constraint equation $250 + 10(\frac{n}{2}) = 250 + 10n - 100$, where n is the number of text messages sent in November.† Constraint equations can often be created and solved by thinking functionally. In the cell phone example, the stated condition is $C(\frac{n}{2}) = C(n) - 100$.

8:2 A pottery factory has two machines: a fast machine and a slow machine. The fast machine paints a pot in 3 min. The slow machine paints a pot in 10 min. Right now there's a pile of 50 unpainted pots waiting to go into the slow machine, and a pile of 28 unpainted pots waiting to go into the fast machine. **(1)** If you start the machines at the same time, which machine will finish its pile first? **(2)** How many min later will the other machine finish its pile? **(3)** Imagine instead that before starting the machines, you move some unpainted pots from the slow machine's pile to the fast machine's pile. How many pots would you move so that the two machines finish painting at the same time?

Answer

(1) The fast machine will finish its pile first. **(2)** The slow machine will finish its pile 416 minutes Later. **(3)** 32 pots.

Click here for a student-facing version of the task.

Refer to the Standards

8.EE.C.7b; MP.1, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Procedural skill and fluency, Application

The two expressions surrounding the equal sign in a one-variable constraint equation in a variable x can be interpreted as defining two functions. If the two functions are denoted respectively by f and g, then the constraint equation reads $f(x) = g(x)$. Solving the equation can then be viewed as finding the value(s) of the input variable for which the two functions have equal output values. One way to solve the equation is therefore to graph the equations $y = f(x)$ and $y = g(x)$ on the same set of coordinate axes, and look for points of intersection of the graphs. The x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$. This approach is often useful for finding solutions approximately, for example by using technology to graph the functions, make tables of values, or find successive approximations. Using technology can also suggest candidates for exact solutions, which can then be checked algebraically.

In part (3) of task 8:2, a condition must be satisfied: two machines must finish painting at the same time. That condition will determine the value of an unknown quantity, the number of pots that must be moved from the slow machine's pile to the fast machine's pile in order for the condition to be met. A way to think about the constraint equation in functional terms could be to define a function $F = 3p$ for the number of minutes for the fast machine to paint p pots, and define another function $S = 10p$ for the number of minutes for the slow machine to paint p pots; then one way to think about the constraint equation is that it reflects the condition $F(28 + x) = S(50 - x)$, where x is the number of pots moved.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: using unit rates; defining a variable and building an expression by calculating with it as if it were a number; and solving multi-step one-variable equations.

Extending the task

How might students drive the conversation further?

- Mathematics is a powerful tool for finding optimal solutions. Students might wonder, or be asked, about how the original piles of 50 pots and 28 pots should be redistributed between the two machines if the goal is to finish all 78 pots *in the minimum possible time*. (Students may be surprised to discover that the answer has already been found. How might they make sense of this finding?)

Additional notes on the design of the task

- Some students may approach the problem by creating and solving a one-variable constraint equation. Some students may create and solve a system of simultaneous two-variable constraint equations. Some students may use technology such as a spreadsheet to create tables and/or graphs that illuminate how the two machines' finishing times depend on the number of pots moved. An important discussion would be for students to find correspondences between different approaches, or for students who used one approach to use a classmate's approach, supported by the classmate's explanations.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:2? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 8:2? In what specific ways do they differ from 8:2?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:2 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

Related Math Milestones tasks

8:9

8:9 A chef is cooking soup in a pot. If the chef keeps the soup gently boiling and doesn't cover the pot, water in the soup will evaporate. As water evaporates away, the soup will get thicker and tastier. Let's use a function equation to model the evaporation process: $D = 12 - 0.1t$. Variable D is the depth of the soup in the pot, in units of cm, and variable t is the amount of time the soup has been boiling, in units of min. **(1)** Graph the function. **(2a)** What is the value of the function for $t = 0$? **(2b)** What does your value in (2a) refer to in the situation? **(2c)** How is the situation at $t = 0$ represented on the graph? **(3)** What is the value of the slope of the graph, and what is the meaning of that value in the situation? **(4)** The soup is ready to eat when its depth is $\frac{2}{3}$ of the initial depth. At what time is the soup ready to eat? **(5)** Is the model useful for knowing what the depth of the soup would be at time $t = 150$ min? Why or why not?

8:7

8:7

City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

8:4

8:4 **(1)** Decide whether each system has exactly one solution, infinitely many solutions, or no solutions. **(2)** For one system, justify your decision to your classmates in two ways: **(a)** drawing graphs of solutions; **(b)** algebraically.

$$\left\{\begin{aligned} y &= \tfrac{2}{3}x + 1 \\ y &= \tfrac{2}{3}x + 2 \end{aligned}\right\} \quad \left\{\begin{aligned} d &= 100 - 4t \\ d &= 3.5 + t \end{aligned}\right\} \quad \left\{\begin{aligned} \tfrac{1}{8}Q + \tfrac{3}{8}R &= -1 \\ Q + 3R &= -8 \end{aligned}\right\}$$

Part (4) of **8:9 Water Evaporation Model** and part (2b) of **8:7 Flight Times and Distances** both involve solving a constraint equation arising from a stated condition. Task **8:4 System Solutions** involves systems of simultaneous two-variable constraint equations; in one or two of the systems, the forms of the equations invite functional thinking.

7:13

7:13 A 15.1-in long wire is bent into the shape of a circle with 2.9 in left over. To the nearest 0.1 in, what is the diameter of the circle?

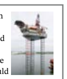
7:14

7:14 Rose and Liba both solved this problem: *Jannat has 4 packs of balloons and 5 single balloons—29 balloons in all. How many balloons are in a pack?* Explain both of Rose's steps. Check that Liba's equations are all true statements about the balloons.

Rose	Liba
$29 - 5 = 24$	Let x be the # of balloons in a pack.
$24 \div 4 = 6$	$4x + 5 = 29$
	$4x = 24$
	$x = 6$

7:8

7:8 In 2018, an oil company rented an oil rig for \$100,000 per day. The company drilled a well and started pumping oil. **(1)** How much oil must be sold each day to equal the rental cost? Note: 42 gal of oil could be sold for \$70 in 2018. **(2)** The company estimates that the profit, P, in millions of dollars, after pumping oil for D days is $P = 0.5D - 40$. **(a)** What is the profit after the first day of pumping oil? **(b)** Make a table of pairs of values (D, P) and graph the ordered pairs. **(c)** How can the company make \$30M of profit? **(3)** An equivalent expression for P is $0.5(D - 80)$. How does the 80 in this expression relate to the company's situation?



6:13

6:13 *Pencils down* Think about the equation $241p = \frac{3}{4}$. Is there a whole number that solves it? Is there a non-whole number that solves it? Convince a classmate that your answers are right.

7:5

7:5 *Pencils down* Think about the equation $x + 4\frac{1}{8} = \frac{2}{3}$. Is there a positive number that solves it? Is there a negative number that solves it? Tell how you decided.

In earlier grades, **7:13 Wire Circle**, **7:14 Comparing Rose and Liba's Solutions**, and part (2c) of **7:8 Oil Business** involve solving a constraint equation arising from a stated condition. Tasks **6:13 Is There a Solution? (Multiplication)** and **7:5 Is There a Solution? (Addition)** emphasize the idea that constraint equations can be interpreted as questions.

† What number does this condition determine yet conceal?

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Anticipating and responding to student thinking about the task

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Language

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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

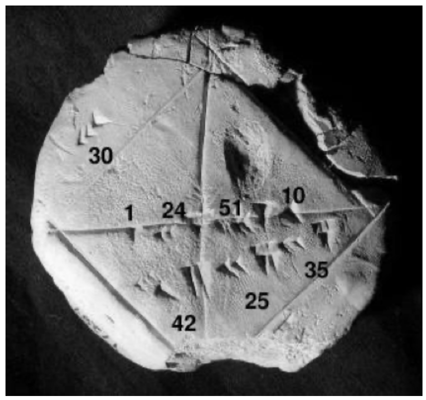
8:3 Bicycle Blueprint

Teacher Notes

Central math concepts

The Pythagorean theorem that extends the concept of number beyond rational numbers has a history long predating the life of Pythagoras himself (c. 570 – c. 495 BC). Special right triangles with three whole-number side lengths were being listed at least as early as c. 1800 BC. Task 8:3 involves a special right triangle with three whole-number side lengths; such triangles are mathematical curiosities, since the square root of a whole number isn't usually a whole number.

The square root of 2 is a more typical case. The figure shows a clay tablet from around 1800–1600 BC that has been interpreted as giving an approximate value of $\sqrt{2}$. Note that in a 45-45-90 right triangle with side lengths measuring 1 unit, the hypotenuse will have length $\sqrt{2}$ units. Rounded to thousandths, $\sqrt{2}$ is approximately 1.414, a slight underestimate because $1.414^2 = 1.9994$. Looking at the pattern of the digits in 1.414, it might be natural to guess that the exact value of $\sqrt{2}$ is the repeating decimal $1.\overline{41}$. That would be equivalent to saying that $\sqrt{2} = \frac{140}{99}$, because $\frac{140}{99}$ in decimal form is $1.\overline{41}$. However, the square of $\frac{140}{99}$ is $\frac{19{,}600}{9{,}801}$ which doesn't equal 2.



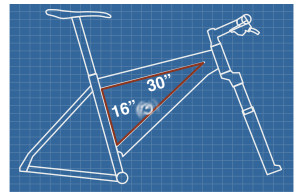
By Bill Casselman - Own work, CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=2154237

The clay tablet in the figure gives a better approximation, $\sqrt{2} \approx 1 + \frac{24}{60} + \frac{51}{3{,}600} + \frac{10}{216{,}000}$ or $\sqrt{2} \approx \frac{305{,}470}{216{,}000}$.

Another fraction approximation, used in India in approximately 800–200 BC, was based on the following rule for calculating the length of the hypotenuse of a 45-45-90 right triangle with given side lengths: "Increase the length [of the side] by its third and this third by its own fourth less the thirty-fourth part of that fourth." In other words, multiply the side by $1 + \frac{1}{3} + \frac{1}{3} \times 4 - \frac{1}{3} \times 4 \times 34 = \frac{577}{408}$. However, the square of $\frac{577}{408}$ is $\frac{332{,}929}{166{,}464}$, which doesn't equal 2. And by around 500 BC, it was known that no fraction equals $\sqrt{2}$.

This also implies that there is no length unit we could choose that would give all three length measures of a 45-45-90 triangle as whole numbers. Still, in practical terms physical measurements always have finite precision. In a blueprint, we would never specify a dimension as $\sqrt{2}$, but rather perhaps as a mixed number of units, such as $1\frac{41}{100}$ inches. Thinking of the unit of measure a hundredths of an inch, $1\frac{41}{100}$ becomes a whole number after all (141 hundredths). In that sense, the triangle in task 8:3 isn't

8:3 On this blueprint for building a bike, part of the bike is shaped like a right triangle. The longest side length is illegible because water spilled on the blueprint. Calculate that side length.



Answer

34"

Click here for a student-facing version of the task.

Refer to the Standards

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Aspect(s) of rigor:

Application

Additional notes on the design of the task

- The task was inspired by a real-life experience (pre-internet) in which a factory blueprint was defective and an employee was requested who knew the Pythagorean theorem.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:3? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 8:3? In what specific ways do they differ from 8:3?

so special after all. Architects, manufacturers, and home crafters alike are working in the end with rational units.

And yet: if we can imagine a square, then we can imagine the diagonal of a square. And if the length of that diagonal is a number, then there have to exist numbers that aren't fractions. A number that isn't a fraction is called irrational—not because the number is unreasonable, but because its value isn't the value of any whole-number ratio.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: understanding the square root symbol; and creating an equation involving the square of an unknown number.

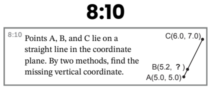
Extending the task

How might students drive the conversation further?

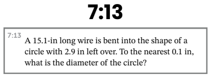
- Students could use the Pythagorean theorem to estimate the percent savings achieved by "cutting a corner" when walking—supposing that the shortcut is the hypotenuse of a right triangle with one side measuring 1 unit and the other side measuring anywhere from 1 to 5 units.

Related Math Milestones tasks

8:10


8:10 Points A, B, and C lie on a straight line in the coordinate plane. By two methods, find the missing vertical coordinate.

7:13


7:13 A 15.1-in long wire is bent into the shape of a circle with 2.9 in left over. To the nearest 0.1 in, what is the diameter of the circle?

The Pythagorean theorem allows distances between points to be calculated when the coordinates of the points are known; as applied to task **8:10 Missing Coordinate**, the theorem could be used to show that segment *AC* is five times as long as segment *AB*. This agrees with the fact that a dilation with center *A* and scale factor 5 takes point *B* to point *C*.

In earlier grades, task **7:13 Wire Circle** involves a decimal approximation to the irrational number π. (As a historical note to compare with the case of $\sqrt{2}$, the number π was proved to be irrational in 1761.)

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:3 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:4 System Solutions

Teacher Notes

Central math concepts

A two-variable equation states a condition which the values of the two variables must satisfy. For example, the equation $x + y = 0$ states a condition on x and y that is satisfied for certain values of x and y (such as $x = 5$ and $y = -5$) but not satisfied for other values of x and y (such as $x = 1$ and $y = 1$). A solution to a two-variable equation is an ordered pair of values—one value for each variable—that makes the equation true when those values are substituted into the equation for the variables.

A two-variable equation for real numbers x and y is called linear when it can be put in the form $Ax + By = C$, where A, B, and C are specific real numbers with A and B not both zero. For example, $2x + 3y = 12$ is a linear equation. The reason for the name "linear equation" is that if all the solutions of the equation (that is, all the ordered pairs of real numbers that satisfy the equation) are plotted as points in the xy coordinate plane, then the resulting set of points is a line. This can be seen by rewriting the equation $2x + 3y = 12$ first in the form $y = -\frac{1}{6}x + 4$, then rewriting it again in the form $\frac{(y - 4)}{(x - 0)} = -\frac{1}{6}$. This last equation is the statement that the slope calculated using the particular point $(0, 4)$ and any other point on the graph is always the same. The condition of constant slope is equivalent to the graph being a straight line.†

Since a two-variable equation states a condition which the values of the two variables must satisfy, a system of two simultaneous equations states two conditions that the values of the variables must satisfy simultaneously. In the case of two linear equations, the graph of the solutions for each equation is a straight line; a point belongs to a given line if and only if the coordinates of the point solve the corresponding equation. A point belongs to both lines simultaneously if and only if the coordinates of the point solve both equations simultaneously. This gives rise to the strategy of solving a system of two simultaneous linear equations in two variables by graphing the solutions of each equation and looking for intersections of the graphs. If an intersection point is found, then the approximate coordinates of the intersection point can be taken for an approximate solution to the system, or the coordinates could be substituted for the variables in both equations to see if they are an exact solution.

Two lines can intersect in 1, 0, or infinitely many points, and a system of two simultaneous linear equations can have 1, 0, or infinitely many solutions. A key factor in the nature of the solutions of a system of two simultaneous linear equations in two variables is whether the corresponding lines are parallel. Non-vertical lines are parallel when their slopes are equal, which can be seen directly from the corresponding equations if they are in $y = mx + b$ form: the lines are parallel when the m coefficient is the same in the two equations. If the equations are in the form $Ax + By = C$, then one can show that the corresponding non-vertical lines are parallel when the ratio A:B is the same for the two equations.

8:4 **(1)** Decide whether each system has exactly one solution, infinitely many solutions, or no solutions. **(2)** For one system, justify your decision to your classmates in two ways: **(a)** drawing graphs of solutions; **(b)** algebraically.

$$\left\{\begin{matrix} y = \frac{2}{3}x + 1 \\ y = \frac{2}{3}x + 2 \end{matrix}\right\} \quad \left\{\begin{matrix} d = 100 - 4t \\ d = 3.5 + t \end{matrix}\right\} \quad \left\{\begin{matrix} \frac{1}{8}Q + \frac{3}{8}R = -1 \\ Q + 3R = -8 \end{matrix}\right\}$$

Answer

(1) The left-hand system has no solutions. The middle system has exactly one solution. The right-hand system has infinitely many solutions. **(2) (a)** Answers may vary depending on which system is chosen. **(b)** Answers may vary depending on which system is chosen and what algebraic arguments are given to justify the decision.

An algebraic justification that the left-hand system has no solutions could involve: (i) substituting, say, $\frac{2}{3}x + 2$ into the first equation for y, and deducing an absurd conclusion; (ii) observing that $y - \frac{2}{3}x$ cannot equal 2 if it equals 1; or another appropriate algebraic argument (see also argument (ii) in the next paragraph).

An algebraic justification that the middle solution has exactly one solution could involve: (i) substituting, say, $3.5 + t$ into the first equation for d, and deducing that $t = 19.3$, $d = 22.8$ is the only possible solution—then checking that $t = 19.3$, $d = 22.8$ does indeed solve the equation; (ii) observing that the right-hand sides of the two equations could define two functions of time, one function that decreases without limit from a large initial value of 100, and another that increases without limit from a small initial value of 3.5—in which case, there is necessarily a moment of time when the value increasing from 3.5 must equal the value decreasing from 100; or another appropriate algebraic argument.

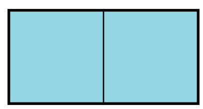
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: using properties of operations to rewrite expressions; analyzing and solving one-variable equations; relating a two-variable equation to the graph of its solutions; and solving equations as a process of reasoning.

Extending the task

How might students drive the conversation further?

- Students could relate the nature of solutions of systems of simultaneous linear equations in two variables to the nature of solutions of single linear equations in one variable by considering the following sequence of word problems:
  - Find the side length of a square if the perimeter is 6 units greater than the side length. (A constraint equation for the problem might be $4x = 6 + x$. This equation could result from function thinking: For a square, the perimeter p is a function of the side length, $p = 4x$. A second function could be defined as $q = x + 6$. For what value of x does $p = q$?)
  - Find the side length of a square if two copies of the square when joined form a rectangle with perimeter equal to six times the side length of the square. (An equation model for the problem might be $x + x + x + x + x + x = 6x$.)

    
  - Find the side length of a square if increasing the side length by 1 unit causes the perimeter to increase by 2 units. (An equation model for the problem might be $4(x + 1) = 4x + 2$.)

  Which problem has 1 solution? 0 solutions? Infinitely many solutions?

Related Math Milestones tasks

8:2

8:2 A pottery factory has two machines: a fast machine and a slow machine. The fast machine paints a pot in 3 min. The slow machine paints a pot in 10 min. Right now there's a pile of 50 unpainted pots waiting to go into the slow machine, and a pile of 28 unpainted pots waiting to go into the fast machine. **(1)** If you start the machines at the same time, which machine will finish its pile first? **(2)** How many min later will the other machine finish its pile? **(3)** Imagine instead that before starting the machines, you move some unpainted pots from the slow machine's pile to the fast machine's pile. How many pots would you move so that the two machines finish painting at the same time?

8:7

8:7

City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

Part (3) of **8:2 Pottery Factory** states a condition that could be expressed as a single-variable equation or as a pair of simultaneous equations analogous in form to the left-hand system or middle system in task 8:4. Part (2b) of **8:7 Flight Times and Distances** involves solving a constraint equation arising from a stated condition.

Answer (continued)

An algebraic justification that the right-hand system has infinitely many solutions could involve: (i) multiplying the first equation through by 8 to show that the two equations are the same (and noting that a single linear equation in two variables has infinitely many solutions); (ii) solving both equations for Q and finding $Q = -8 - 3R$ in both cases, so that all the system requires is a free choice of value for R followed by a calculation of the required corresponding value of Q; or another appropriate algebraic argument.

Click here for a student-facing version of the task.

Refer to the Standards

8.EE.C.8; MP.5, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Procedural skill and fluency

Additional notes on the design of the task

- Both equations in the left-hand system have "$y = mx + b$" form, which may invite a graphing approach for students practiced in translating between graphs and equations in this form.
- Both equations in the middle system have a "Quantity = initial value + (rate)·(time)" form, which may invite a functional interpretation of the algebra.

7:13

7:13 A 15.1-in long wire is bent into the shape of a circle with 2.9 in left over. To the nearest 0.1 in, what is the diameter of the circle?

7:14

7:14 Rose and Liba both solved this problem: *Jannat has 4 packs of balloons and 5 single balloons—29 balloons in all. How many balloons are in a pack?* Explain both of Rose's steps. Check that Liba's equations are all true statements about the balloons.

Rose	**Liba**
$29 - 5 = 24$	Let x be the # of balloons in a pack.
$24 \div 4 = 6$	$4x + 5 = 29$
	$4x = 24$
	$x = 6$

7:8

7:8 In 2018, an oil company rented an oil rig for $100,000 per day. The company drilled a well and started pumping oil. **(1)** How much oil must be sold each day to equal the rental cost? Note: 42 gal of oil could be sold for $70 in 2018. **(2)** The company estimates that the profit, P, in millions of dollars, after pumping oil for D days is $P = 0.5D - 40$. **(a)** What is the profit after the first day of pumping oil? **(b)** Make a table of pairs of values (D, P) and graph the ordered pairs. **(c)** How can the company make $30M of profit? **(3)** An equivalent expression for P is $0.5(D - 80)$. How does the 80 in this expression relate to the company's situation?

6:13

6:13 *Pencils down* Think about the equation $241p = \frac{3}{4}$. Is there a whole number that solves it? Is there a non-whole number that solves it? Convince a classmate that your answers are right.

7:5

7:5 *Pencils down* Think about the equation $x + 4\frac{1}{8} = \frac{2}{3}$. Is there a positive number that solves it? Is there a negative number that solves it? Tell how you decided.

In earlier grades, **7:13 Wire Circle**, **7:14 Comparing Rose and Liba's Solutions**, and part (2c) of **7:8 Oil Business** involve solving a constraint equation arising from a stated condition. Tasks **6:13 Is There a Solution? (Multiplication)** and **7:5 Is There a Solution? (Addition)** emphasize the idea that constraint equations can be interpreted as questions.

Additional notes on the design of the task (continued)

- The equations in the right-hand system are intended to have conspicuous numbers 3 and 8 so as to prompt looking for and making use of structure.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:4? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 8:4? In what specific ways do they differ from 8:4?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:4 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† In the case $A = 0$, the linear equation $Ax + By = C$ is equivalent to the equation $y = C/B$, the solutions of which graph as a horizontal line; in the case $B = 0$, then the linear equation $Ax + By = C$ is equivalent to the equation $x = C/A$, the solutions of which graph as a vertical line.

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Anticipating and responding to student thinking about the task

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Language

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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
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8:5 Rotations Preserve Angle Measure

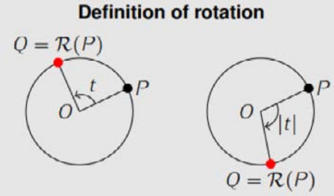
Teacher Notes

Central math concepts

Some students may have experiences with touch screen displays in or out of school when using electronic devices for such purposes as map navigation, internet browsing, playing video games, or editing photos. Interacting with touch screens could involve dragging an object, "flipping an image left-right," rotating a map, or pinching-to-zoom on a photo. These operations are reminiscent of the translations, reflections, rotations, and dilations that students study in geometry and that are the basis for careful definitions of congruence and similarity.

Rotations in particular are defined as shown in the figure, which shows the effect of a rotation around the point *O* through an angle of measure $t°$ in two cases, $t \geq 0$ and $t < 0$. In particular, in the case $t > 0$, the rotation takes point *P* clockwise to point *Q*, which is located the same distance from *O* as point *P* and forms angle $\angle POQ$ with measure $t°$.†

Definition of rotation



The rotation $\mathcal{R}$ around the point O through the angle t takes a point P to the point $Q = \mathcal{R}(P)$ as follows. If $P = O$, then $\mathcal{R}(O) = O$. If $P \neq O$ and $t \geqslant 0$, then Q is on the circle with center O and radius $|OP|$ so that $\angle POQ = t°$ and Q is counterclockwise from P. If $t < 0$, we rotate clockwise by $|t|°$.

Image from: Common Core Standards Writing Team. (2016, March 24). Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry. Tucson, AZ: Institute for Mathematics and Education, University of Arizona, p. 14.

Rotations, along with reflections and translations, are rigid motions. The following properties of rigid motions are assumed as axioms:

1. Rigid motions map lines to lines, rays to rays, and segments to segments.
2. Rigid motions preserve distance.
3. Rigid motions preserve angle measure.

The definition of a rotation emphasizes that a rotation through a nonzero angle moves all points of the plane other than the center of rotation. Therefore a rotation doesn't just move a triangle, or a line segment, or other figure; rather, the rotation moves the points of the plane, and because geometric figures are made of those moving points, the figures move too. As the Geometry Progression document notes,

> When the transformation is a rigid motion (a translation, rotation, or reflection) it is useful to represent it using transparencies because two copies of the plane are represented, one by the piece of paper and one by the transparency. These correspond to the domain and range of the transformation, and emphasize that

8:5 Using physical models, transparencies, or geometry software, illustrate the fact that *rotations take angles to angles of the same measure.*

Answer

Answers may vary but illustrations should identify both that rotations take angles to angles, and that rotations take angles to angles of the same measure.

Click here for a student-facing version of the task.

Refer to the Standards

8.G.A.1; MP.3, MP.5, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- Because it is an axiom that rotations preserve angle, the intent of task 8:5 isn't to have students prove that rotations preserve angle. Rather, by illustrating the fact, students build a base of experience for reasoning with the axioms.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:5? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 8:5? In what specific ways do they differ from 8:5?

the transformation acts on the entire plane, taking each point to another point. The fact that rigid motions preserve distance and angle is clearly represented because the transparency is not torn or distorted. (p. 14)

In high school, students will learn more explicitly that transformations are functions from the plane to itself.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: applying transformations; practicing geometry notation; using definitions; and constructing mathematical arguments.

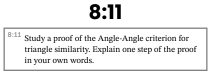
Extending the task

How might students drive the conversation further?

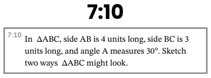
- Students could draw consequences from the properties of rotations. For example, if a square with area 100 square units is rotated, what is the area of the rotated square? Justify your answer carefully using properties of rotations.

Related Math Milestones tasks

8:11


8:11 Study a proof of the Angle-Angle criterion for triangle similarity. Explain one step of the proof in your own words.

7:10


7:10 In ΔABC, side AB is 4 units long, side BC is 3 units long, and angle A measures 30°. Sketch two ways ΔABC might look.

Task **8:11 Angle-Angle Similarity Proof** concerns a proof that (in the most general case) relies upon the angle-measure preserving property of rotations.

In earlier grades, task **7:10 Triangle Conditions** involves conditions under which one, more than one, or no triangle may be constructed; these considerations underlie the triangle congruence theorems proved using transformations.

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:5 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† The sense of a rotation is sometimes characterized by a "right-hand rule": for a rotation through a positive angle, if you place the edge of your right hand along segment *OP* with your thumb pointing up from the paper, your fingers will sweep from *P* to *Q*.

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8:5 Rotations Preserve Angle Measure

Teacher Notes

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

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Solution Paths

- What solution paths might you expect to see?
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Central math concepts

Part (1). Every fraction $\frac{a}{b}$ can be expressed as a decimal that terminates or eventually repeats, and conversely every terminating or repeating decimal can be expressed as a fraction. To find the decimal expansion of a fraction $\frac{a}{b}$, one could perform the long division $a \div b$. For example, $\frac{7}{12} = 7 \div 12$, and if we perform the long division as shown, we eventually find that the process repeats. The result of the process could be expressed as

$$\frac{7}{12} = 0.58\overline{3}.$$

```
      .58333...
  12⟌70000000
      60
      100
       96
        40
        36
         40
         36
          40
          36
           40
            ⋮
```

Looking at the structure of the decimal $0.58\overline{3}$, it appears to be the sum of 0.58 and $0.00\overline{3}$, which is to say the sum of $\frac{58}{100}$ and $\frac{1}{100}$ of $\frac{1}{3}$. Let's see if that fraction sum indeed equals $\frac{7}{12}$:

$$\frac{58}{100} + \frac{1}{100} \times \frac{1}{3}$$
$$= \frac{58}{100} + \frac{1}{300}$$
$$= \frac{174}{300} + \frac{1}{300}$$
$$= \frac{175}{300}$$
$$= \frac{7}{12}.$$

Thus we have traveled full circle from the fraction to the decimal back to the fraction.

Similarly, in part (a) of task 8:6, the number $1.041\overline{6}$ could be seen as

$$1 + \frac{4}{100} + 0.001\overline{6}$$
$$1 + \frac{4}{100} + \frac{1}{6} \div 100$$
$$1 + \frac{4}{100} + \frac{1}{600}$$
$$1 + \frac{24}{600} + \frac{1}{600}$$
$$1 + \frac{25}{600}$$
$$1 + \frac{1}{24}$$
$$\frac{25}{24}.$$

8:6 Write as a fraction in lowest terms: **(1)** $1.041\overline{6}$. **(2)** $3^2 \cdot 3^{-5}$.

Answer

(1) $\frac{25}{24}$. **(2)** $\frac{1}{27}$.

Click here for a student-facing version of the task.

Refer to the Standards

8.NS.A.1, 8.EE.A.1; MP.6, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- There are many ways to approach part (a) of the task. Students could experiment with a calculator to decide that $1000 \times 0.000\overline{6} = \frac{2}{3}$ or recognize that $0.000\overline{6} = \frac{2}{3} \div 1000$.
- Whatever the approach, carrying out the long division will settle the question of whether $\frac{25}{24}$ equals $1.041\overline{6}$.
- The intent of part (a) is to involve students in thinking synthetically about fractions, decimals, place value, and their relationships; it is not a procedural fluency task. (Part (b) could be considered a procedural fluency task after the properties of integer exponents are understood.)

Part (2). Students extend the meaning of exponents from positive exponents to integer exponents based on the fundamental property $\boldsymbol{x^a x^b = x^{a+b}}$. This property summarizes the patterns in positive exponents that students are familiar with from previous grades, for example $7^3\, 7^2 = (7{\cdot}7{\cdot}7)(7{\cdot}7) = 7^5$ and $7^2 = 7^{1+1} = 7^1{\cdot}7^1 = 7{\cdot}7$. The property also gives meaning to negative exponents and the exponent 0. For example, consider the question of what we take 7^0 to be. Suppose initially we believe we should take 7^0 to be 0. The fundamental property has something to say about that choice, because it would follow that

$$7^2 = \overbrace{7^{2+0} = 7^2\, 7^0}^{\text{Using the property here}} = 7^2 \cdot 0 = 0.$$

Thus if we take 7^0 to be zero, we will also have to take 7^2 to be zero. But the familiar meaning of 7^2 is $7{\cdot}7 = 49$, so taking 7^0 to be 0 wouldn't be using the fundamental property to extend familiar meanings of exponents.

If we aren't going to take 7^0 to be zero, then could we take it to be something else? Perhaps 7^0 could be any positive number we like? To find out what's possible, let's apply the fundamental property again:

$7^0\, 7^0 = 7^{0+0}$ (using the property with both exponents zero)

$7^0\, 7^0 = 7^0$. (because 0 + 0 = 0)

Dividing both sides of the last equation by 7^0 (remember that 7^0 is nonzero), we find

$7^0 = 1$.

So when using the fundamental principle to extend the meaning of exponents from positive to integer exponents, there is only one possibility for the value of 7^0, which is 1. And the "7" here could have been any nonzero number, so expressing the conclusion generally, x^0 equals 1 for any nonzero number x. This consequence of the fundamental property is sometimes called "the zero power rule."

Another consequence of the fundamental property is sometimes called "the negative exponent rule," $x^{-a} = \frac{1}{x^a}$. This rule follows from the fundamental property by applying the fundamental property with equal and opposite exponents: $x^a x^{-a} = x^{a+-a} = x^0$. Remembering that $x^0 = 1$, we have $x^a x^{-a} = 1$. This says that x^a and x^{-a} are reciprocals, or equivalently that $x^{-a} = \frac{1}{x^a}$. That's "the negative exponent rule." Students working from long lists of integer exponent rules may not realize that those rules are special-purpose applications of a fundamental property, rather than independent principles.

For more observations on integer exponents, the relationship between fractions and repeating decimals, and the beginnings of students' studies of irrational numbers, see the relevant *Progression* document.†

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:6? Locate 2–3 similar tasks in that unit. How are the tasks similar to each other, and to 8:6? In what specific ways do they differ from 8:6?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:6 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: adding fractions; remembering fraction-decimal equivalents; understanding place value; practicing long division in simple cases; and calculating mentally.

Extending the task

How might students drive the conversation further?

- Students could check the answer $\frac{25}{24}$ by calculating 25 ÷ 24 by long division.
- If $d = 0.\overline{6}$ then students could evaluate the claim that $10d = 6 + d$. Taking the equation as true, what do you get if you solve it for d?
- Students could repeat the above with $d = 0.\overline{9}$.

Related Math Milestones tasks

8:4

8:4 **(1)** Decide whether each system has exactly one solution, infinitely many solutions, or no solutions. **(2)** For one system, justify your decision to your classmates in two ways: **(a)** drawing graphs of solutions; **(b)** algebraically.

$\begin{Bmatrix} y = \frac{2}{3}x + 1 \\ y = \frac{2}{3}x + 2 \end{Bmatrix}$ $\begin{Bmatrix} d = 100 - 4t \\ d = 3.5 + t \end{Bmatrix}$ $\begin{Bmatrix} \frac{1}{8}Q + \frac{3}{8}R = -1 \\ Q + 3R = -8 \end{Bmatrix}$

8:7

8:7

City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

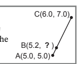
(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

8:9

8:9 A chef is cooking soup in a pot. If the chef keeps the soup gently boiling and doesn't cover the pot, water in the soup will evaporate. As water evaporates away, the soup will get thicker and tastier. Let's use a function equation to model the evaporation process: $D = 12 - 0.1t$. Variable D is the depth of the soup in the pot, in units of cm, and variable t is the amount of time the soup has been boiling, in units of min. **(1)** Graph the function. **(2a)** What is the value of the function for $t = 0$? **(2b)** What does your value in (2a) refer to in the situation? **(2c)** How is the situation at $t = 0$ represented on the graph? **(3)** What is the value of the slope of the graph, and what is the meaning of that value in the situation? **(4)** The soup is ready to eat when its depth is $\frac{2}{3}$ of the initial depth. At what time is the soup ready to eat? **(5)** Is the model useful for knowing what the depth of the soup would be at time $t = 150$ min? Why or why not?

8:10

8:10 Points A, B, and C lie on a straight line in the coordinate plane. By two methods, find the missing vertical coordinate.



7:9

7:9 **(1)** alculate. **(a)** $-4.1 + 4$ **(b)** $5 \div (-6)$ **(c)** $1(-1 - 1)$ **(d)** $2 - (-\frac{1}{2})$ **(e)** $(-\frac{3}{8})(-8)$ **(f)** $0 - \frac{1}{3}$ **(g)** $\frac{1}{7.9} * 7.9$ **(h)** $(\frac{1}{2} - \frac{1}{4})(-9 + 9)$.

(2) Show calculation 1(a) on a number line.

Fractions and decimals appear in several grade 8 tasks including **8:4 System Solutions**, **8:7 Flight TImes and Distances**, **8:9 Water Evaporation Model**, and **8:10 Missing Coordinate**.

In earlier grades, task **7:9 Calculating with Rational Numbers** extends calculation from positive fractions to signed rational numbers.

† Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6–8, The Number System; High School, Number*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:7 Flight Times and Distances

Teacher Notes

Central math concepts

In this task the central concept is the linear function. Linear functions are a generalization of proportional relationships. To see this more clearly, let's first look closely at proportional relationships.

In a proportional relationship, two quantities x and y covary in such a way that the ratio y/x is constant. Proportional relationships are functions that have the equation form $y = rx$. If y is proportional to x, then something more is also true, namely, changes in y are proportional to changes in x. This follows from the distributive property, because if y_1 and y_2 are two different y values, then $y_1 = rx_1$ and $y_2 = rx_2$, so the change in quantity y equals

$$y_2 - y_1$$
$$= rx_2 - rx_1$$
$$= r(x_2 - x_1)$$

In other words, the change in y is always a multiple r of the change in x. Another way to say that changes in y are proportional to changes in x is to say that *the rate of change of* y *with respect to* x *is a constant value*. (The constant value is r.)

There are more functions besides proportional relationships in which the rate of change of y with respect to x is constant. In any function that has a function equation of the form $y = mx + b$, changes in y are directly proportional to changes in x, even though y is not proportional to x when b is nonzero. This again follows from the distributive property, because if y_1 and y_2 are two different y values, then $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, so the change in quantity y equals

$$y_2 - y_1$$
$$= (mx_2 + b) - (mx_1 + b)$$
$$= mx_2 - mx_1$$
$$= m(x_2 - x_1)$$

In other words, the change in y is always a constant multiple m of the change in x. Another way to say that changes in y are proportional to changes in x is to say that *the rate of change of* y *with respect to* x *is a constant value.* (The constant value is m.)

In a situation where one quantity changes at a constant rate with respect to another quantity, we can express the first quantity as a linear function of the second, even if the first quantity isn't directly proportional to the second quantity. In task 8:7, the flight times shown in the table change at a constant rate with respect to city-to-city distances; this can be seen from the fact that the values in the second column increase by 0.2 hours for every increase of 100 miles in the values in the first column. This is why there is a linear function that models the data in the table.

For more discussion of these concepts, see pp. 5, 6 of *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions* (Common Core Standards Writing Team, March 1, 2013. Tucson, AZ: Institute for Mathematics and Education, University of Arizona) and pp. 11, 12 of *Progressions for the Common Core State Standards in*

8:7

City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

Answer

(1) $H = 0.002D + 0.6$, where H is the number of hours of flight time and D is the city-to-city distance in miles. Any function equation that is equivalent to $y = 0.002x + 0.6$ is correct, as long as the equation is consistent with the contextual meanings of both variables and the meanings of both variables are stated explicitly. Examples: $y = (0.2/100)(x - 200) + 1$ is correct, if y is stated to be the number of hours of flight time and x is stated to be the city-to-city distance in miles; $y = 12x + 36$ is correct, if y is the number of minutes of flight time and x is the city-to-city distance in hundreds of miles. **(2) (a)** $H = 0.002(1000) + 0.6 = 2.6$ hr. **(b)** $2 = 0.002D + 0.6 \Rightarrow D = 700$ mi. (Variable names, equations, and units may differ depending on the form of the function equation in part (1). Note, the task requires creating and solving equations - not just calculations with numbers.) **(3)** Answers vary; click here for an example.

Click here for a student-facing version of the task.

Refer to the Standards

8.F; MP.1, MP.2, MP.4, MP.7, MP.8. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with

[Mathematics (draft). Grades 6–8, Expressions and Equations](#) (*Ibid*., 22 April 2011).

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: calculating unit rates; graphing ordered pairs from a table of values; converting units; creating equations to describe relationships between quantities that covary; solving one-variable linear equations to find an unknown value; and using technology.

Extending the task

How might students drive the conversation further?

- Students could ask or be asked about the meaning of the constants 0.002 and 0.6 in the function equation $H = 0.002D + 0.6$. The coefficient 0.002 is the number of hours taken to fly 1 mile; this value is much less than 1, because planes fly so quickly (it takes much less than 1 hour to fly 1 mile). Meanwhile, the constant 0.6 indicates a flight time of 0.6 hours for a trip of 0 miles; one interpretation of this value would be as a quantity of time that is wasted on the two runways. Be aware that students may be unfamiliar with air travel in their own life experiences or in the movies they have seen, stories they have read, and so on.
- In a situation where one quantity is a linear function of another, it is also true that the second quantity is a linear function of the first. This is because $\Delta y = m\Delta x$ implies $\Delta x = (\frac{1}{m})\Delta y$. Students could ask or be asked about how the city-to-city distance depends on flight time. That linear function can be written $y = 500x - 300$, where y is the city-to-city distance in miles and x is the flight time in hours. Students could discuss the meaning of the coefficient 500 in the context of the situation (it has units of miles per hour and could be interpreted as flight speed).
- Students could calculate the average speed for flights of different distances, observing that the average speed increases for longer trips, because the "startup cost" of 0.6 hours becomes a smaller and smaller fraction of the total trip time.

Related Math Milestones tasks

8:1

8:1 Xavier's assignment for science class was to write notes to summarize a chapter in his textbook. At 4:45 p.m., he had 12 pages left to summarize. At 6:00 p.m., he had 7 pages left. Assuming a linear model, about how many more hours will it take him to finish summarizing?

6:4

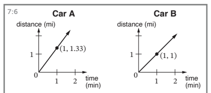
6:4 My car drives 570 mi with 15 gal of gas. **(1)** *Mental math/Pencil and paper* **(a)** If I drive 57 mi, I'll use ___ gal. **(b)** If I drive 5,700 mi, I'll use ___ gal. **(c)** If I have 5 gal left, I can drive ___ more mi. **(d)** I can drive ___ mi with 30 gal. **(2)** *Calculator* Calculate both unit rates for the proportional relationship. **(3)** **(a)** If I drive 532 mi, I'll use ___ gal. **(b)** If I have 11 gal left, I can drive ___ more mi. **(4)** Make a two-column table using your answers to (1a), (1c), (1d), (3a), and (3b). Then use graph paper to plot the values in the table. Label the axes of your plot.

8:9

8:9 A chef is cooking soup in a pot. If the chef keeps the soup gently boiling and doesn't cover the pot, water in the soup will evaporate. As water evaporates away, the soup will get thicker and tastier. Let's use a function equation to model the evaporation process: $D = 12 - 0.1t$. Variable D is the depth of the soup in the pot, in units of cm, and variable t is the amount of time the soup has been boiling, in units of min. **(1)** Graph the function. **(2a)** What is the value of the function for $t = 0$? **(2b)** What does your value in (2a) refer to in the situation? **(2c)** How is the situation at $t = 0$ represented on the graph? **(3)** What is the value of the slope of the graph, and what is the meaning of that value in the situation? **(4)** The soup is ready to eat when its depth is $\frac{2}{3}$ of the initial depth. At what time is the soup ready to eat? **(5)** Is the model useful for knowing what the depth of the soup would be at time $t = 150$ min? Why or why not?

7:6

7:6



Car A and Car B were moving at constant speed, as shown in the graphs. **(1)** At the end of the first minute, how many miles had each car moved? **(2)** Which car was moving faster? **(3)** For the faster car, write a formula for the number of miles moved in n minutes. **(4)** How many miles does the faster car move in 10 minutes?

6:6

6:6 A farmer uses a tractor to plant corn quickly in the springtime. The farmer plants 216 acres every 12 hours. Create a formula for the number of acres the farmer plants in n hours.

Tasks **8:1 Xavier's Notes** and **8:9 Water Evaporation Model** prominently feature linear models.

In earlier grades, tasks **7:6 Car A and Car B**, **6:6 Planting Corn**, and **6:4 Gas Mileage** prominently feature proportional relationships.

Refer to the Standards (continued)

locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

- In part (1), students create a function equation. In part (2), students can use the function equation to answer two questions about particular flight times and distances. If one wished, one could use the function equation to answer many more questions about particular flight times and distances; for that reason, one might say that a function equation contains in a single statement all the infinitely many specific possibilities inherent in a situation with covarying quantities. Part (3) shows the power of a function equation in combination with technology to quickly generate dozens, hundreds, even thousands of those particular possibilities. This power is one reason why it is worth learning how to develop function equations to model situations, instead of only ever solving problems about one unknown value in a particular occurrence of a situation.

Curriculum connection

- In which unit of your curriculum would you expect to find tasks like 8:7? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 8:7? In what specific ways do they differ from 8:7?
- Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:7 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
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- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:8 Heart Rate and Exercise

Teacher Notes

Math Milestones™

Central math concepts

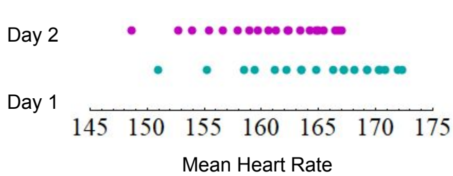
When interpreting a scatter plot, the first order of business is to understand what fact is being represented by a single (x, y) data point. What is the story behind one data point? What process of measurement or experimentation gave rise to the number x and the number y together? It is premature to evaluate trends or patterns in the distribution as a whole without knowing the meaning of a single data point. In task 8:8 for example, the data points were produced during an experiment in exercise physiology:

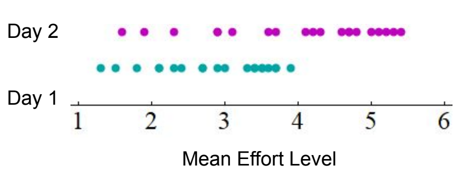
> A researcher asked people doing exercise to rate their effort level. The researcher also measured people's heart rates. Data were taken on two different days. Each person's heart rate (beats per min.) and effort (1–6 scale) were recorded every 3 min. A group average was then calculated, creating one data point such as (150.9, 1.3).

So the data point (150.9, 1.3) refers to the fact that at a certain moment of time during the training session, the average heart rate among the people doing exercise was 150.9 beats per minute, while the average effort level rating on a 1–6 scale was 1.3 at that same point in time.

Note that because there are 20 data points, and because a new data point was produced every three minutes, one can infer that the data set for each day was the outcome of a 60-minute training session. Apparently, as the training session wore on, people's heart rates increased and so did their subjective level of effort. Qualitatively that was true on both days, but quantitatively that pattern played out differently on Day 1 vs. Day 2.

The circumstance that links the values 150.9 and 1.3 into an ordered pair is that the two measurements were taken at the same time. The values 150.9 and 1.3 are paired into a single observation (150.9, 1.3). However, it is possible and sometimes valuable to "intentionally forget" that the x values and the y values in a data set are paired, by examining the distribution of x values and the distribution of y values separately (see the figure).





In the upper figure, the heart rate data for each day has been plotted on a stacked dot plot. The upper figure shows that mean heart rates were slightly lower on Day 2 (see CCSS 7.SP.B.3). In the lower figure, the effort

8:8 A researcher asked people doing exercise to rate their effort level. The researcher also measured people's heart rates. Data were taken on two different days. **(1)** Use technology to plot the data from both days. (View heart rates in a window from 145 to 175.) Describe the main patterns you see. **(2)** On one of the days, the exercise room was warm, and on the other day, the room was cool. Which day do you think was the warm day? Tell how you decided, and support your answer with calculations.

Heart Rate & Effort in Exercise

Day 1 HR, Effort	Day 2 HR, Effort
150.9, 1.3	148.6, 1.6
155.2, 1.5	152.7, 1.9
158.5, 1.8	153.9, 2.3
159.4, 2.1	155.4, 2.9
161.2, 2.1	156.6, 2.9
162.2, 2.3	157.9, 3.1
163.5, 2.4	158.9, 3.6
163.5, 2.7	159.7, 3.7
164.8, 2.7	160.6, 4.1
166.3, 2.9	161.3, 4.2
167.2, 3.0	162.3, 4.3
167.2, 3.3	162.4, 4.6
168.1, 3.4	163.4, 4.7
169.2, 3.4	164.2, 4.8
169.2, 3.5	164.8, 4.7
170.3, 3.5	165.0, 5.0
170.8, 3.6	165.4, 5.1
170.4, 3.7	167.0, 5.2
171.9, 3.7	166.5, 5.3
172.3, 3.9	166.7, 5.4

Click here to get the data online. Each person's heart rate (beats per min.) and effort (1–6 scale) were recorded every 3 min. A group average was then calculated, creating one data point such as (150.9, 1.3).

Answer

(1) See figure. (A spreadsheet of the data is online here.) **(2)** The data suggest that Day 2 was the warm day and Day 1 was the cool day. Explanations may vary but could include the following observation(s): (i) When it's warmer, you can feel like you're working harder even if your body is doing the same thing. And even though heart rates were a bit lower on Day 2 (average heart rate about 161 bpm vs. 165 bpm on Day 1), people still felt they were working harder on Day 2 (average effort level about 4, vs. 2.8 on Day 1). (ii) A linear model for Day 2 has a steeper slope than a linear model for Day 1. That means on Day 2, raising your heart rate required a greater increase in effort. This again suggests that Day 2 was the warm day.

level data for each day has been plotted on a stacked dot plot. The lower figure shows that mean effort levels were generally higher on Day 2. This is also visible on the scatter plot—not as a steeper slope for Day 2, but rather in the fact that the Day 2 data points are generally shifted vertically upward compared to the Day 1 data points.

The slope of a graph has several important interpretations that recur over and over in applications. One interpretation of slope is geometric: slope measures the steepness of a line in the coordinate plane. Another interpretation of slope is that the value of the slope equals the rate of change for a quantity that varies with time. A third interpretation of slope is as a marginal return: a slope of m means that each unit increase in quantity x corresponds to (or costs, or yields) an increase of amount m in quantity y. (The "increase" is a decrease when m is negative.) In the context of task 8:8, each unit increase in heart rate "costs" an increase in effort. The marginal cost of that increase was greater on the warmer day, statistically speaking.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: thinking about data in context; calculating measures of center and measures of variation; describing patterns in distributions of univariate and bivariate measurement data; and using technology.

Extending the task

How might students drive the conversation further?

- Students could use technology to fit a best-fit line to both data sets and interpret the two slopes in context.
- Students could create a similar experiment of their own to measure some aspect of exercise and the body's responses to exertion, and ask ons about the da

Related Math Milestones tasks

8:1

8:1 Xavier's assignment for science class was to write notes to summarize a chapter in his textbook. At 4:45 p.m., he had 12 pages left to summarize. At 6:00 p.m., he had 7 pages left. Assuming a linear model, about how many more hours will it take him to finish summarizing?

6:3

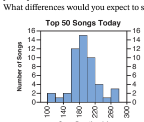
6:3 The table shows temperatures at the South Pole before and after midnight on October 10–11, 2019.

Time	Hours after Midnight	Temp °F
8:00 pm	–4	–42
9:00 pm	–3	–42
10:00 pm	–2	–41
11:00 pm	–1	–40
Midnight	0	–39
1:00 am	1	–39
2:00 am	2	–38

Plot the data on graph paper and label the plot. Describe any patterns you see.

6:7

6:7 **(1)** Look up the 50 top songs on a music streaming service. Type each song's duration into a spreadsheet. **(2)** Write a sentence about the data giving a measure of center and a measure of variability. **(3)** Make a histogram of the data.* **(4)** Write a sentence describing the overall pattern of the distribution and any striking deviations from the overall pattern. **(5)** Imagine that one year from now, you go back online and repeat (1)–(4). In what ways would you expect the data distribution to look similar? What differences would you expect to see?



*Use this histogram for (4) and (5) if you don't do (3).

Task **8:1 Xavier's Notes** involves given information that could be viewed as a two-point data set of bivariate data.

In earlier grades, task **6:3 South Pole Temperatures** uses the coordinate plane to represent a set of bivariate measurement data, while task **6:7 Song Length Distribution** involves a distribution of univariate data.

Answer (continued)

Click here for a student-facing version of the task.

Refer to the Standards

8.SP.A.1–3; MP.4, MP.5. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Additional notes on the design of the task

- The data for task 8:8 comes from a 2004 undergraduate research study in the field of exercise physiology.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:8? Locate 2–3 similar tasks in that unit. How are the tasks you found similar to each other, and to 8:8? In what specific ways do they differ from 8:8?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:8 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

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Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

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Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:9 Water Evaporation Model

Teacher Notes

Central math concepts

Initial value and rate of change. Properties of a function can be read from the function equation that defines it. For a function equation of the form $F = c + rt$, two important properties of function F can be read from the equation: its initial value and its rate of change. The initial value is c, as can be seen by evaluating the function at $t = 0$ ($F = c + r{\cdot}0 = c$). To see that the rate of change of the function is r, let's evaluate the function at two arbitrary different times, t_1 and t_2, which leads to corresponding output values $c + rt_1$ and $c + rt_2$. The rate of change of the function is the difference of those two function values divided by the difference in the two times:

$$\text{Rate} = \frac{\text{change in } F}{\text{change in } t} = \frac{(c + rt_2) - (c + rt_1)}{t_2 - t_1}.$$

This simplifies:

$$\frac{c + rt_2 - c - rt_1}{t_2 - t_1} = \frac{rt_2 - rt_1}{t_2 - t_1} = \frac{r(t_2 - t_1)}{t_2 - t_1} = r.$$

Thus, for a function defined by $F = c + rt$, the function's rate of change is r, independently of the interval of time over which the rate of change is calculated. A function defined by a function equation of the form $F = c + rt$ could therefore be called a constant-rate function.

Graph. The graph of a function defined by $F = c + rt$ is the set of all ordered pairs $(t, c + rt)$. These ordered pairs "pair" outputs with inputs. When plotted on a coordinate plane, the ordered pairs create a picture of how quantity F depends on quantity t. Something interesting happens when you calculate the slope of the graph of a constant-rate function $F = c + rt$. Using two arbitrary distinct points on the graph, $(t_1, c + rt_1)$ and $(t_2, c + rt_2)$, the slope can be calculated as

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{(c + rt_2) - (c + rt_1)}{t_2 - t_1} = r$$

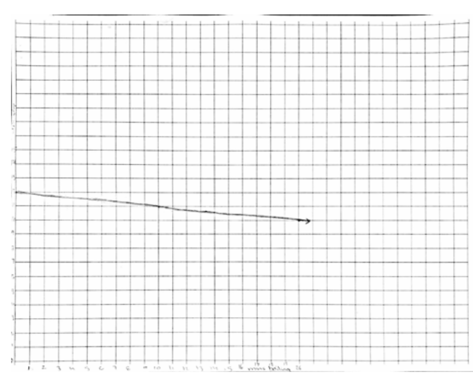
(thanks to the same algebra as before). This shows that the slope of the graph has value r, independently of the two points chosen to calculate it. Not only is the slope of the graph equal to the rate of change of the function, but also the constancy of the slope of the graph implies that the graph of a constant-rate function is a straight line. This justifies our practice of calling constant-rate functions linear.

"The slope of the graph equals the rate of change" is a principle that generalizes far beyond the middle grades. Differential calculus extends this idea to nonlinear functions, for which the slope and rate of change both vary with time.

8:9 A chef is cooking soup in a pot. If the chef keeps the soup gently boiling and doesn't cover the pot, water in the soup will evaporate. As water evaporates away, the soup will get thicker and tastier. Let's use a function equation to model the evaporation process: $D = 12 - 0.1t$. Variable D is the depth of the soup in the pot, in units of cm, and variable t is the amount of time the soup has been boiling, in units of min. **(1)** Graph the function. **(2a)** What is the value of the function for $t = 0$? **(2b)** What does your value in (2a) refer to in the situation? **(2c)** How is the situation at $t = 0$ represented on the graph? **(3)** What is the value of the slope of the graph, and what is the meaning of that value in the situation? **(4)** The soup is ready to eat when its depth is $\frac{2}{3}$ of the initial depth. At what time is the soup ready to eat? **(5)** Is the model useful for knowing what the depth of the soup would be at time $t = 150$ min? Why or why not?

Answer

(1) See figure. **(2)** **(a)** 12. **(b)** The value 12 refers to the depth of the soup in the pot in centimeters just when it begins to boil. **(c)** The graph includes the point (0, 12), and this point represents the fact that the soup in the pot is 12 cm deep just when it begins to boil. **(3)** The slope of the graph is -0.1, which means that every minute, the depth of the soup in the pot decreases by 0.1 cm. **(4)** The soup is ready to eat after it has been boiling for 40 min. **(5)** The model isn't useful for knowing what the depth of the soup would be at time t = 150 min, because the model says that the depth of the soup is -3 cm at that time, but the depth of the soup cannot be negative.



Click here for a student-facing version of the task.

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: graphing a function defined by a function equation; interpreting and using negative numbers in context; working with simple fractions and decimals in context; creating and solving a constraint equation to find an unknown value; and using technology.

Extending the task

How might students drive the conversation further?

- The function equation $D = 12 - 0.1t$ can be solved for $t = 100(12 - D)$. Students could be asked to interpret this function equation: what is the input variable, what is the output variable, and can we make sense of the rule defined by the expression on the right-hand side?
- Supposing that the chef forgets about the soup and leaves it on the stove for 150 minutes, students could sketch a graph showing the depth of the soup from $t = 0$ to $t = 150$ min. This graph would then itself be a mathematical model that applies to the situation over a longer period of time than the function model stated in task 8:9. Students could debate whether the least possible value of the depth variable is reasonably 0 cm (which would imply an empty pot) or reasonably something greater than 0 cm (corresponding to a layer of burnt soup-sludge).

Related Math Milestones tasks

8:1

8:1 Xavier's assignment for science class was to write notes to summarize a chapter in his textbook. At 4:45 p.m., he had 12 pages left to summarize. At 6:00 p.m., he had 7 pages left. Assuming a linear model, about how many more hours will it take him to finish summarizing?

8:7

8:7

City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

Tasks **8:1 Xavier's Notes** and **8:7 Flight Times and Distances** prominently feature linear models.

7:12

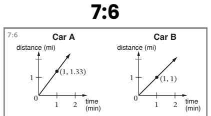
7:12 In 1972 in Loma, Montana, the temperature changed from −54°F to +49°F in a 24-hr period. Calculate the average rate at which the temperature changed. Answer to the nearest tenth in units of degrees/hr.

7:8

7:8 In 2018, an oil company rented an oil rig for $100,000 per day. The company drilled a well and started pumping oil. **(1)** How much oil must be sold each day to equal the rental cost? Note: 42 gal of oil could be sold for $70 in 2018. **(2)** The company estimates that the profit, P, in millions of dollars, after pumping oil for D days is $P = 0.5D - 40$. **(a)** What is the profit after the first day of pumping oil? **(b)** Make a table of pairs of values (D, P) and graph the ordered pairs. **(c)** How can the company make $30M of profit? **(3)** An equivalent expression for P is $0.5(D - 80)$. How does the 80 in this expression relate to the company's situation?

7:6

7:6



Car A and Car B were moving at constant speed, as shown in the graphs. **(1)** At the end of the first minute, how many miles had each car moved? **(2)** Which car was moving faster? **(3)** For the faster car, write a formula for the number of miles moved in n minutes. **(4)** How many miles does the faster car move in 10 minutes?

6:6

6:6 A farmer uses a tractor to plant corn quickly in the springtime. The farmer plants 216 acres every 12 hours. Create a formula for the number of acres the farmer plants in n hours.

6:4

6:4 My car drives 570 mi with 15 gal of gas. **(1)** *Mental math/Pencil and paper* **(a)** If I drive 57 mi, I'll use ___ gal. **(b)** If I drive 5,700 mi, I'll use ___ gal. **(c)** If I have 5 gal left, I can drive ___ more mi. **(d)** I can drive ___ mi with 30 gal. **(2)** *Calculator* Calculate both unit rates for the proportional relationship. **(3)** **(a)** If I drive 532 mi, I'll use ___ gal. **(b)** If I have 11 gal left, I can drive ___ more mi. **(4)** Make a two-column table using your answers to (1a), (1c), (1d), (3a), and (3b). Then use graph paper to plot the values in the table. Label the axes of your plot.

In earlier grades, task **7:12 Temperature Change** involves a time-varying quantity that has a negative rate of change. Tasks **7:6 Car A and Car B**, **6:6 Planting Corn**, and **6:4 Gas Mileage** prominently feature proportional relationships. Task **7:8 Oil Business** involves an algebraic expression that defines a linear function.

Refer to the Standards

8.F; MP.1, MP.2, MP.4, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

- Part (5) is about being aware of the limitations and hidden assumptions behind mathematical models. See the section on "How might students drive the conversation further?"

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:9? Locate 2-3 similar tasks in that unit. How are the tasks you found similar to each other, and to 8:9? In what specific ways do they differ from 8:9?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:9 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

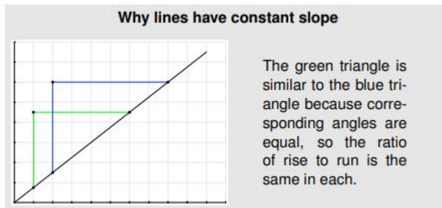
- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:10 Missing Coordinate

Teacher Notes

Central math concepts

This task is most directly about the connections between proportional relationships and non-vertical lines in the coordinate plane. With regard to those connections, the most essential fact is that **the slope of such a line has the same value regardless of which two points are chosen to calculate it.** (See the figure.)



The ratio of rise to run is the same for the green and red triangles because the triangles are similar (by the angle-angle criterion for triangle similarity). *Image from*: Common Core Standards Writing Team. (2011). *Progressions for the Common Core State Standards for Mathematics: 6–8, Expressions and Equations* (Draft, 4/22/2011), p. 12. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

The fact that the slope has the same value regardless of which two points are used to calculate it could be used directly as one method for determining the missing coordinate in task 8:10, by writing and solving an equation such as $\frac{(y - 5)}{(5.2 - 5)} = \frac{(7 - 5)}{(6 - 5)}$. Related reasoning might involve completing the figure by drawing two right triangles (which are similar) and making use of the equal side ratios.

Another connection to proportional relationships would involve reasoning that since 5.2 is "20% of the way from 5 to 6," the missing *y*-coordinate should be "20% of the way from 5 to 7." And since 20% of a doubly long distance must be doubly long, the increment from 5 to 5.2 in the *x*-coordinate will be matched by an increment from 5 to 5.4 in the *y*-coordinate. (0.4 is twice as much as 0.2.)

Equations aren't necessary for solving task 8:10, but students could use the equation of a line to find the missing coordinate. Indeed, the fact that the slope has the same value regardless of which two points are used to calculate it underlies all the work students do with the equation of a line. To see this, imagine calculating the slope of a non-vertical line as follows. First, choose a variable point on the line with coordinates (x, y). Next, choose a fixed-but-arbitrary point and denote its coordinates by (x_1, y_1). Then we can calculate two lengths, $y - y_1$ and $x - x_1$. As long as $x \neq x_1$, we can calculate the slope as the ratio of these two lengths:

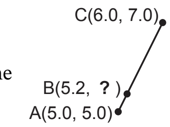
$$\frac{\text{rise}}{\text{run}} = \frac{y - y_1}{x - x_1}$$

Because the value of this ratio will be independent of the chosen points, we have

$$\frac{y - y_1}{x - x_1} = m$$

where m is a number that doesn't depend on y, y_1, x, or x_1, but only depends on the line itself. This is one form of the equation of the line, though perhaps not the form most commonly used. Using algebra, we can rewrite the equation in many different ways, such as $y - y_1 = m(x - x_1)$, $y = mx + (y_1 - mx_1)$, or $y = mx + b$.

8:10 Points A, B, and C lie on a straight line in the coordinate plane. By two methods, find the missing vertical coordinate.



Answer

5.4 (methods may vary).

Click here for a student-facing version of the task.

Refer to the Standards

8.EE.B; MP.1, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts, Application

Additional notes on the design of the task

- The lack of gridlines in the diagram is intentional in order to encourage working with the numbers directly instead of counting grid units. However, students who are having trouble getting started might be advised to transfer the diagram carefully to a coordinate grid, or to carefully draw some gridlines on the diagram.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:10? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 8:10? In what specific ways do they differ from 8:10?

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: determining horizontal and vertical distances between points in the coordinate plane; finding an unknown side length in a triangle, given all side lengths in a triangle known to be similar; finding lengths in a scale drawing; and simple mental calculations involving decimal numbers.

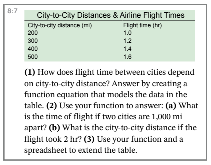
Extending the task

How might students drive the conversation further?

- If students have worked with geometric transformations, they could translate points A, B, and C so that A moves to the origin. This corresponds to subtracting 5 from every coordinate value in the problem. Then the coordinates of the transformed points are (0, 0), (0.2, ?), and (1, 2). The transformed problem may be easier to solve for the coordinates (0.2, 0.4). Now translating back to the original positions by adding 5 to every coordinate value, point B in its original position will have coordinates (5.2, 5.4). Transforming a problem to make it easier to solve is one of the most powerful mathematical practices of all.
- If some students used the two-point form of the equation of a line to determine the equation of the line through points A and C, $y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1) \Rightarrow y = 2x - 5$, while other students used slope reasoning or similar triangles reasoning to develop an equation for the unknown coordinate, such as $\frac{(y - 5)}{(5.2 - 5)} = \frac{2}{1}$, then students could discuss correspondences between the two approaches. For example, instead of solving the equation by the fastest route as $\frac{(y - 5)}{(0.2)} = 2 \Rightarrow y - 5 = 0.4 \Rightarrow y = 5.4$, suppose we solved it this way: $\frac{(y - 5)}{(5.2 - 5)} = \frac{2}{1} \Rightarrow y - 5 = 2(5.2 - 5) \Rightarrow y = 2(5.2) - 10 + 5 \Rightarrow y = 2(5.2) - 5$. This last equation matches up with the equation of the line, $y = 2x - 5$, where $x = 5.2$.

Related Math Milestones tasks

8:7

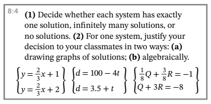

8:7

City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

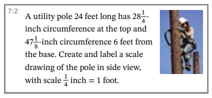
8:4


8:4

(1) Decide whether each system has exactly one solution, infinitely many solutions, or no solutions. **(2)** For one system, justify your decision to your classmates in two ways: **(a)** drawing graphs of solutions; **(b)** algebraically.

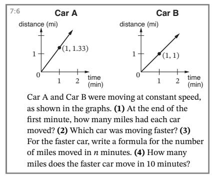
$$\begin{Bmatrix} y = \frac{2}{3}x + 1 \\ y = \frac{2}{3}x + 2 \end{Bmatrix} \begin{Bmatrix} d = 100 - 4t \\ d = 3.5 + t \end{Bmatrix} \begin{Bmatrix} \frac{1}{8}Q + \frac{3}{8}R = -1 \\ Q + 3R = -8 \end{Bmatrix}$$

7:2


7:2

A utility pole 24 feet long has $28\frac{1}{4}$-inch circumference at the top and $47\frac{1}{8}$-inch circumference 6 feet from the base. Create and label a scale drawing of the pole in side view, with scale $\frac{1}{4}$ inch = 1 foot.

7:6


7:6

Car A and Car B were moving at constant speed, as shown in the graphs. **(1)** At the end of the first minute, how many miles had each car moved? **(2)** Which car was moving faster? **(3)** For the faster car, write a formula for the number of miles moved in n minutes. **(4)** How many miles does the faster car move in 10 minutes?

Math Milestones task **8:7 Flight Times and Distances** presents pairs of values that fall along a straight line when graphed. (The constant slope in this context can be interpreted as a speed.) Math Milestones task **8:4 System Solutions** involves two-variable equations.

In earlier grades, Math Milestones task **7:2 Utility Pole Scale Drawing** involves creating a scale drawing, and Math Milestones task **7:6 Car A and Car B** involves proportional relationships that graph as straight lines.

Curriculum connection (continued)

2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:10 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

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Solution Paths

- What solution paths might you expect to see?
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Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
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Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:11 Angle-Angle Similarity Proof

Teacher Notes

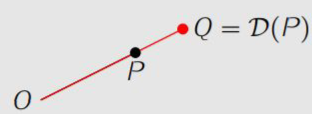
Math Milestones™

Central math concepts

Some students may have experiences with touch screen displays in or out of school when using electronic devices for such purposes as map navigation, internet browsing, playing video games, or editing photos. Interacting with touch screens could involve dragging an object, "flipping an image left-right," rotating a map, or pinching-to-zoom on a photo. These operations are reminiscent of the translations, reflections, rotations, and dilations that students study in geometry and that are the basis for careful definitions of congruence and similarity.

In particular, two figures are similar if they can be made to coincide by a sequence of rigid transformations and dilations. By definition, a dilation with center *O* and positive scale factor *r* takes points other than *O* to points that are *r* times as far from *O* as they originally were. The figure shows the effect of a dilation with center *O* and scale factor $r > 1$. The dilation takes point *P* to point *Q*, which is located *r* times as far away from *O* as point *P*, in the direction defined by ray *OP*. By the looks of the figure, the scale factor of this dilation is approximately $r \approx 1.5$.

Definition of dilation



The dilation $\mathcal{D}$ with center O and positive scale factor r leaves O unchanged and takes every point P to the point $Q = \mathcal{D}(P)$ on the ray OP whose distance from O is $r|OP|$.

Image from: Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona, p. 16.

Similarity links shape to proportionality. A dilation with positive scale factor *r* will take a segment *AB* of length $|AB|$ to a dilated segment with length $r|AB|$. If two segments have a certain length ratio, then the dilated segments will have the same length ratio. This also guarantees that dilations preserve angle measure. More generally, shape is preserved under dilations, but size is not preserved unless the scale factor equals 1.

The Angle-Angle criterion for triangle similarity is the statement that if two angles of a triangle are congruent to two angles of another triangle, then the two triangles are similar. To prove the Angle-Angle criterion using transformations, one examines a figure showing two triangles that are arbitrary except for having two congruent angles, and one describes a sequence of rigid transformations and dilations that place one triangle directly on top of the other.

Students can apply the similarity criterion to understand the concept of the slope of a line. The slope of a non-vertical line in the coordinate plane is the same when calculated between any two distinct points, which is a consequence of the line's straightness. The role of geometry in defining

8:11 Study a proof of the Angle-Angle criterion for triangle similarity. Explain one step of the proof in your own words.

Answer

Answers may vary, depending on the proof chosen and the step of the proof chosen. The resource page for task 8:11 has a link to a proof that could be used. Note: the discussion in this Teacher Note is based on geometric transformations (CCSS 8.G.A), but a Euclidean proof could be used for task 8:11.

Click here for a student-facing version of the task.

Refer to the Standards

8.G.A.5; MP.3, MP.7. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Concepts

Additional notes on the design of the task

- The task does not require that students construct a proof, only that they examine a proof to understand and explain a step of the argument.
- The resource page can provide a link to a proof of the Angle-Angle similarity criterion. Different proofs could be used, and suggestions are welcome for additional proofs to add to the resource page.

slope is described in the *Progression* document for 7–8 and High School Geometry† and in the *Progression* document for 6–8 Expressions and Equations‡ (see also the Teacher Note for **8:10 Missing Coordinate**).

Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: applying transformations; using ratio thinking; practicing geometry notation; using definitions; and constructing mathematical arguments.

Extending the task

How might students drive the conversation further?

- Students could realize that if triangle *ABC* and another triangle *DEF* have two congruent angles, then in fact they must have three congruent angles, because the measures of the angles of a triangle sum to 180°.
- Because of this, the Angle-Angle criterion for triangle similarity could have been called the Angle-Angle-Angle criterion for triangle similarity. What might be a reason to prefer using a criterion that refers to two angles?
- Students could decide whether two triangles that are congruent will sometimes, always, or never satisfy the Angle-Angle criterion for triangle similarity.

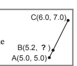
Related Math Milestones tasks

8:5

8:5 Using physical models, transparencies, or geometry software, illustrate the fact that *rotations take angles to angles of the same measure.*

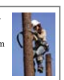
8:10

8:10 Points A, B, and C lie on a straight line in the coordinate plane. By two methods, find the missing vertical coordinate.



7:2

7:2 A utility pole 24 feet long has $28\frac{1}{4}$-inch circumference at the top and $47\frac{1}{8}$-inch circumference 6 feet from the base. Create and label a scale drawing of the pole in side view, with scale $\frac{1}{4}$ inch = 1 foot.



Task **8:5 Rotations Preserve Angle Measure** focuses on a property of rotations that figures into the logic of an angle-angle similarity argument. Task **8:10 Missing Coordinate** offers the opportunity to connect slope to similarity.

In earlier grades, task **7:2 Utility Pole Scale Drawing** implicitly involves similarity and explicitly involves length ratios.

Curriculum connection

1. In which unit of your curriculum would you expect to find tasks like 8:11? Locate 2-3 similar tasks in that unit. How are the tasks similar to each other, and to 8:11? In what specific ways do they differ from 8:11?
2. Thinking about the curriculum unit you identified, at what point in the unit might a task like 8:11 help students converge toward grade-level thinking about the important mathematics in the task? What factors would you consider in choosing when to use such a task in the unit?*

† Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

‡ Common Core Standards Writing Team. (2011). *Progressions for the Common Core State Standards for Mathematics: 6–8, Expressions and Equations* (Draft, 4/22/2011), p. 12. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

* Math Milestones™ tasks are not designed for summative assessment. Used formatively, the tasks can reveal and promote student thinking.

Anticipating and responding to student thinking about the task

Imagine how students might think about the task, and what you might see and hear while they work.

On this page, you can write your thoughts on the following questions.

Solution Paths

- What solution paths might you expect to see?
- What representations might you see? What correspondences between those representations might be noticed by students (or be worth pointing out to students) and discussed by them?
- What misconceptions or partial understandings might be revealed as students work on the task? How could you respond to these positively and productively?

Language

- What might you expect to hear from students engaged with the task? What does that language reveal about their mathematical thinking, and how might you respond to different ways of thinking?
- If students are using early English or using multiple languages in an integrated communication system, how might you help their classmates see those mathematical ideas as valuable?
- Even when using nascent language, students are thinking and communicating their thinking. What might it look like to respond positively and productively to the mathematics in their thinking before giving feedback on the language used?

Identity, Agency, and Belonging

- How can you engage students' interests, experiences, or funds of knowledge?
- How can you build students' self-confidence as learners, thinkers, and doers of mathematics?
- What choices are there for a student to make in the task? How can you build students' agency to the point where they notice and make these choices to solve problems?
- How might one use feedback to build student agency? Where might there be opportunities to build students' self-confidence?

8:12 Fish Tank Design

Teacher Notes

Central math concepts

A common form of mathematical modeling is using mathematics to create a design that satisfies a set of real-world constraints. The constraints in a design problem might arise from finite resources, space requirements, basic physics, even customer preferences. In some cases, the goal of the mathematical model is to find an optimal solution given the constraints. In other cases, any design that satisfies the constraints is acceptable.

Modeling tasks frequently make use of mathematics that was first learned in previous grades, such as unit rates in the case of task 8:12. Mass density is a kind of unit rate; the mass density of water is approximately 1,000 when measured in units of kilograms per cubic meter. Multiplying a mass density in kilograms per cubic meter by a volume in cubic meters results in a mass in kilograms, just as multiplying a speed in meters per second by a time in seconds results in a distance in meters. Other useful examples of density include area densities such as the population density of a city (measured in persons per square mile or persons per square kilometer) or the intensity of sunlight: the solar power flowing into each square meter of a solar cell can be as high as 1,000 watts.

Modeling typically involves a high degree of student agency, because there are more choices for students to make in such tasks, and because the necessary mathematical tools aren't conveniently assembled for the job beforehand. In task 8:12 for example, volume formulas aren't given. Students can use the internet to find formulas they need for geometric measures, and/or they can reason that the problem of finding the volume of a cylinder amounts to finding the area of its base and multiplying that area by the cylinder height. But even with a cylinder formula in hand, the task is designed so that students will still need agency to devise a way to calculate the volume of a quarter-cylinder wedge shape. For that matter, the word "volume" is nowhere stated in the task. Rather, the unit m^3 is a signal about the key quantity; modeling often involves noticing and reasoning about the units of measure in a situation.

For more information and resources about mathematical modeling, see the Modeling section in the digital Coherence Map on achievethecore.org.

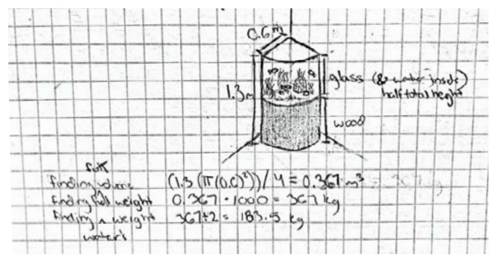
Relevant prior knowledge

The following mathematics knowledge may be activated, extended, and deepened while students work on the task: drawing to scale; converting between meters and customary units of length units (feet and/or inches); working with compound units such as m^3; using unit rates; and applying the formula for volume of a cylinder.

8:12 Design a fish tank that fits into the corner of a room. Use a quarter of a cylinder as a model for the tank. To share your design, make a diagram showing the tank measurements. Also, calculate the weight of the water when your tank is filled (1 m^3 of water weighs about 1,000 kg). Write your calculation steps so that a classmate could understand how you did it.

Answer

Answers may vary. The figure shows an example solution by an 8th-grade student. The student began by making a mistake (taking the height of the water to be 1.3 m, twice the actual height) but then corrected the mistake in the final step (dividing by two).



Click here for a student-facing version of the task.

Refer to the Standards

8.G.C.9; MP.4, MP.5. Standards codes refer to www.corestandards.org. One purpose of the codes is that they may allow a task to shed light on the Standards cited for that task. Conversely, reading the cited Standards may suggest opportunities to extend a task or draw out its implications. Finally, Standards codes may also assist with locating relevant sections in curriculum materials, including materials aligned to comparable standards.

Aspect(s) of rigor:

Application

Extending the task

How might students drive the conversation further?

- Students could use algebra to show connections between the formulas they used: for example, $\frac{1}{4}V_{\text{cylinder}} = \frac{1}{4}(\pi r^2 h) = (\frac{1}{4}\pi r^2)h = (\frac{1}{4}\text{ circle area})\cdot(\text{height})$.
- Students could go online for conversions necessary to calculate how many gallons of water will be in the tank they designed.
- Students could extend the design process to include designing a base for the tank.
- Students could go online to find out more about the hobby and profession of being an aquarist.

Related Math Milestones tasks

8:1

8:1 Xavier's assignment for science class was to write notes to summarize a chapter in his textbook. At 4:45 p.m., he had 12 pages left to summarize. At 6:00 p.m., he had 7 pages left. Assuming a linear model, about how many more hours will it take him to finish summarizing?

8:2

8:2 A pottery factory has two machines: a fast machine and a slow machine. The fast machine paints a pot in 3 min. The slow machine paints a pot in 10 min. Right now there's a pile of 50 unpainted pots waiting to go into the slow machine, and a pile of 28 unpainted pots waiting to go into the fast machine. **(1)** If you start the machines at the same time, which machine will finish its pile first? **(2)** How many min later will the other machine finish its pile? **(3)** Imagine instead that before starting the machines, you move some unpainted pots from the slow machine's pile to the fast machine's pile. How many pots would you move so that the two machines finish painting at the same time?

8:7

8:7 City-to-City Distances & Airline Flight Times

City-to-city distance (mi)	Flight time (hr)
200	1.0
300	1.2
400	1.4
500	1.6

(1) How does flight time between cities depend on city-to-city distance? Answer by creating a function equation that models the data in the table. **(2)** Use your function to answer: **(a)** What is the time of flight if two cities are 1,000 mi apart? **(b)** What is the city-to-city distance if the flight took 2 hr? **(3)** Use your function and a spreadsheet to extend the table.

8:9

8:9 A chef is cooking soup in a pot. If the chef keeps the soup gently boiling and doesn't cover the pot, water in the soup will evaporate. As water evaporates away, the soup will get thicker and tastier. Let's use a function equation to model the evaporation process: $D = 12 - 0.1t$. Variable D is the depth of the soup in the pot, in units of cm, and variable t is the amount of time the soup has been boiling, in units of min. **(1)** Graph the function. **(2a)** What is the value of the function for $t = 0$? **(2b)** What does your value in (2a) refer to in the situation? **(2c)** How is the situation at $t = 0$ represented on the graph? **(3)** What is the value of the slope of the graph, and what is the meaning of that value in the situation? **(4)** The soup is ready to eat when its depth is $\frac{2}{3}$ of the initial depth. At what time is the soup ready to eat? **(5)** Is the model useful for knowing what the depth of the soup would be at time $t = 150$ min? Why or why not?

7:2

7:2 A utility pole 24 feet long has $28\frac{1}{4}$-inch circumference at the top and $47\frac{1}{8}$-inch circumference 6 feet from the base. Create and label a scale drawing of the pole in side view, with scale $\frac{1}{4}$ inch = 1 foot.

Applied unit rates are implicit or explicit in tasks **8:1 Xavier's Notes**, **8:2 Pottery Factory**, **8:7 Flight Times and Distances**, and **8:9 Water Evaporation Model**.

In earlier grades, task **7:2 Utility Pole Scale Drawing** involves geometric measures and scaling.

Additional notes on the design of the task

- The first modeling challenge in the task is to visualize the shape of a "quarter cylinder." This involves both reading comprehension (inferring the shape from the phrase "fits in the corner") and agency (because imagining the shape is up to the student).
- People can sometimes be surprised by the heaviness of a quantity of water. A cube of water measuring 1 m on a side weighs a thousand kilograms, or over 2,000 pounds. It's very possible to design a tank that would be too heavy for its base, which is why the task requires a calculation of the weight of the water in the tank. (In practice, owners of large fish tanks are sometimes advised to position the tank over two floor joists.)
- The task specifies that students "write your calculation steps so that a classmate could understand how you did it." This is a simple version of the "Report" step of the Modeling cycle. (When a modeling task involves optimizing or decision-making, the Report step could be a more substantial work-product such as a typed document or slide presentation.)

Curriculum connection

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Teacher Notes

Math Milestones™

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